CHAPTER III.

TRIGONOMETRIC SERIES.

A. TRIGONOMETRIC FUNCTIONS.

66. For the engineer, and especially the electrical engineer, a perfect familiarity with the trigonometric functions and trigonometric formulas is almost as essential as familiarity with the multiplication table. To use trigonometric methods efficiently, it is not sufficient to understand trigonometric formulas enough to be able to look them up when required, but they must be learned by heart, and in both directions; that is, an expression similar to the left side of a trigonometric formula must immediately recall the right side, and an expression similar to the right side must immediately recall the left side of the formula.

Trigonometric functions are defined on the circle, and on the right triangle.

Let in the circle, Fig. 28, the direction to the right and upward be considered as positive, to the left and downward as negative, and the angle α be counted from the positive horizontal $\overline{OA}$, counterclockwise as positive, clockwise as negative.

The projector s of the angle α, divided by the radius, is called $\sin\alpha$; the projection c of the angle α, divided by the radius, is called $\cos\alpha$.

The intercept t on the vertical tangent at the origin A, divided by the radius, is called $\tan\alpha$; the intercept ct on the horizontal tangent at B, or 90 deg ., behind A, divided by the radius, is called $\cot\alpha$.

- Thus, in Fig. 28,

$$\left.\begin{array}{l}\sin\alpha=\frac{s}{r};\quad\cos\alpha=\frac{c}{r}\\ \tan\alpha=\frac{t}{r};\quad\cot\alpha=\frac{ct}{r}\end{array}\right\}\tag{1}$$

In the right triangle, Fig. 29, with the angles α and β, opposite respectively to the cathetes a and b, and with the hypotenuse c, the trigonometric functions are:

$$\left.\begin{array}{r}\sin\alpha=\cos\beta=\frac{a}{c};\quad\cos\alpha=\sin\beta=\frac{b}{c}\\ \tan\alpha=\cot\beta=\frac{a}{b};\quad\cot\alpha=\tan\beta=\frac{b}{a}.\end{array}\right\}.\tag{2}$$

By the right triangle, only functions of angles up to 90 deg ., or $\frac{\pi}{2}$, can be defined, while by the circle the trigonometric functions of any angle are given. Both representations thus must be so familiar to the engineer that he can see the trigo-

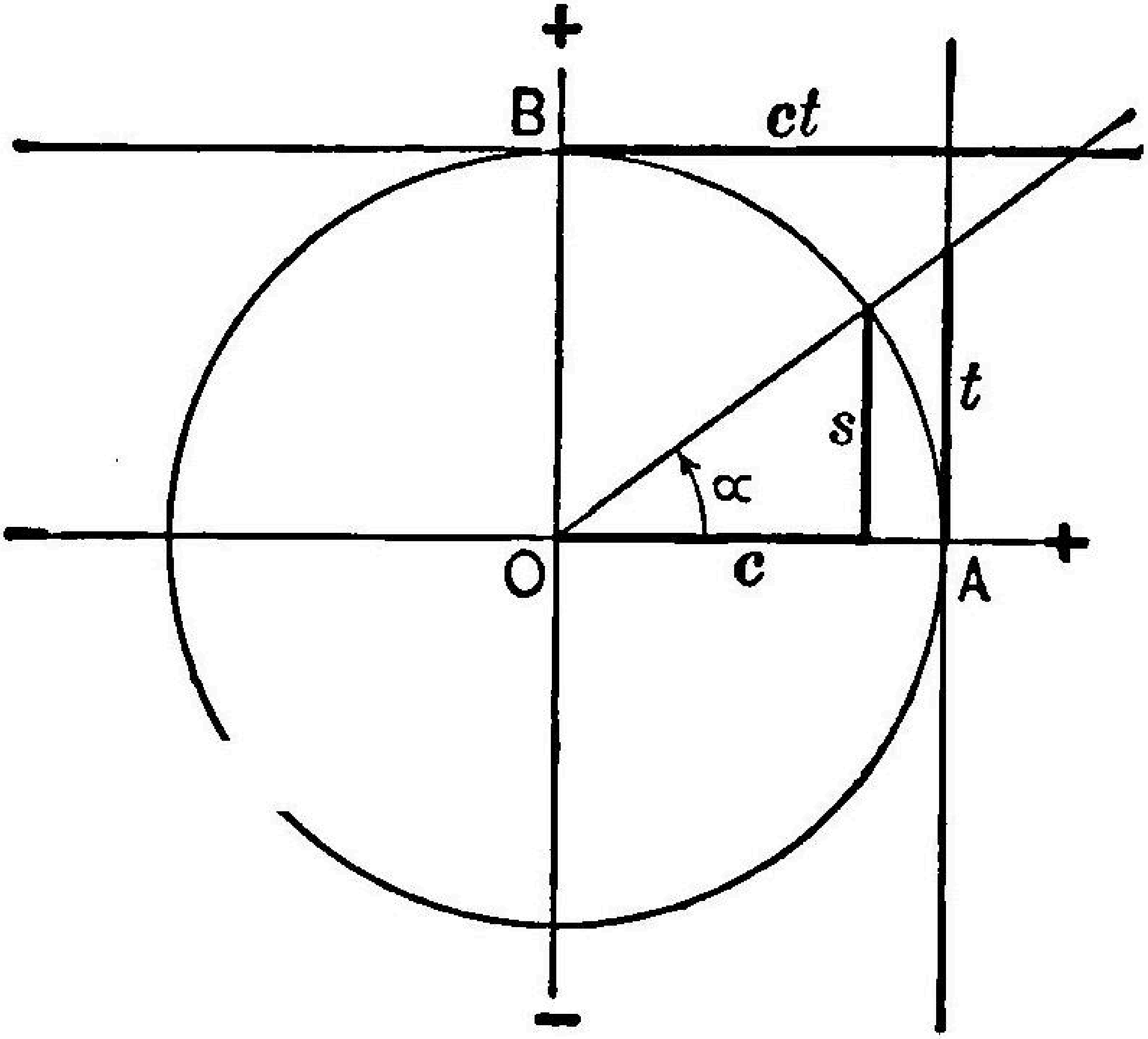

Fig. 28. Circular Trigonometric Functions.

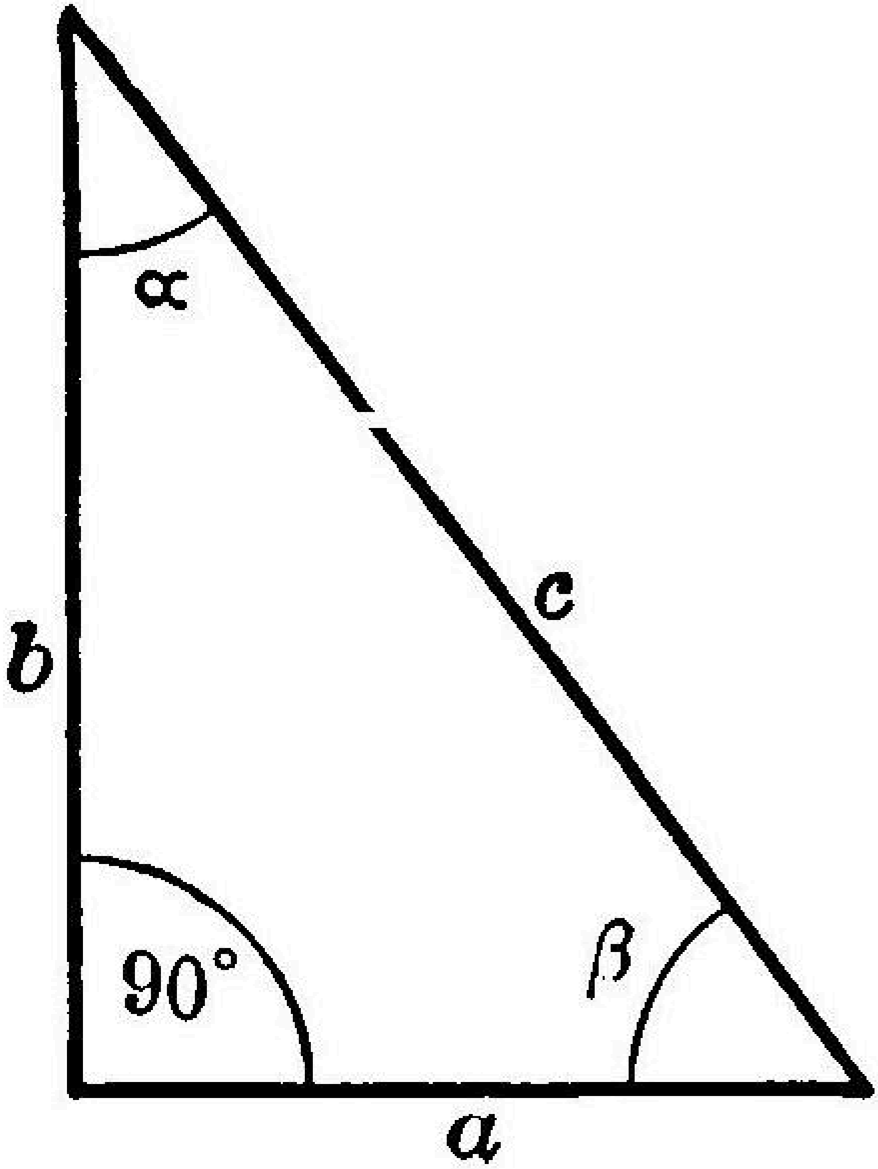

Fig. 29. Triangular Trigonometric Functions.

nometric functions and their variations with a change of the angle, and in most cases their numerical values, from the mental picture of the diagram.

67. Signs of Functions. In the first quadrant, Fig. 28, all trigonometric functions are positive.

In the second quadrant, Fig. 30, the $\sin\alpha$ is still positive, as s is in the upward direction, but $\cos\alpha$ is negative, since c is toward the left, and $\tan\alpha$ and $\cot\alpha$ also are negative, as t is downward, and ct toward the left.

In the third quadrant, Fig. 31, $\sin\alpha$ and $\cos\alpha$ are both negative: s being downward, c toward the left; but $\tan\alpha$ and $\cot\alpha$ are again positive, as seen from t and ct in Fig. 31.

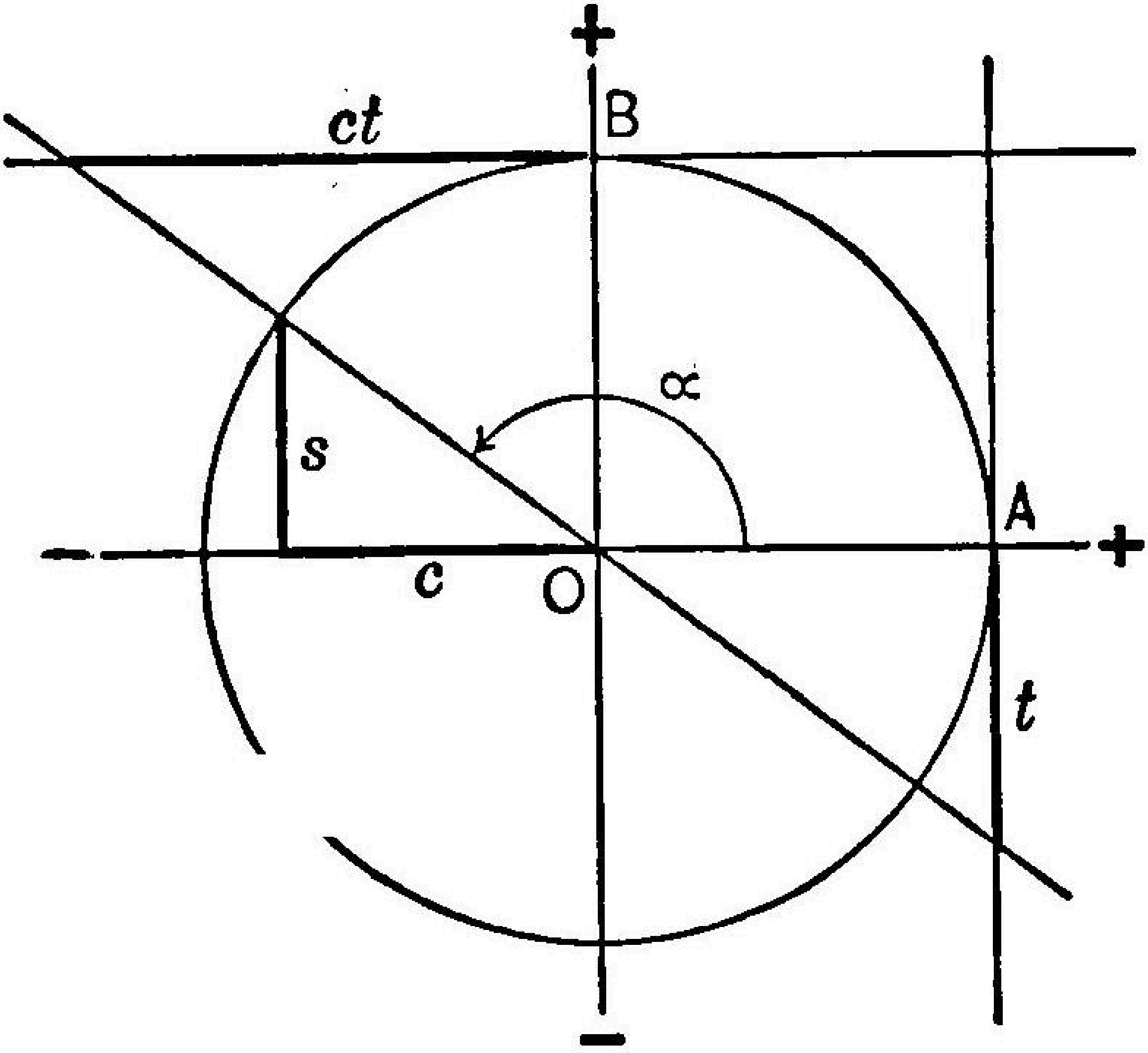

Fig. 30. Second Quadrant.

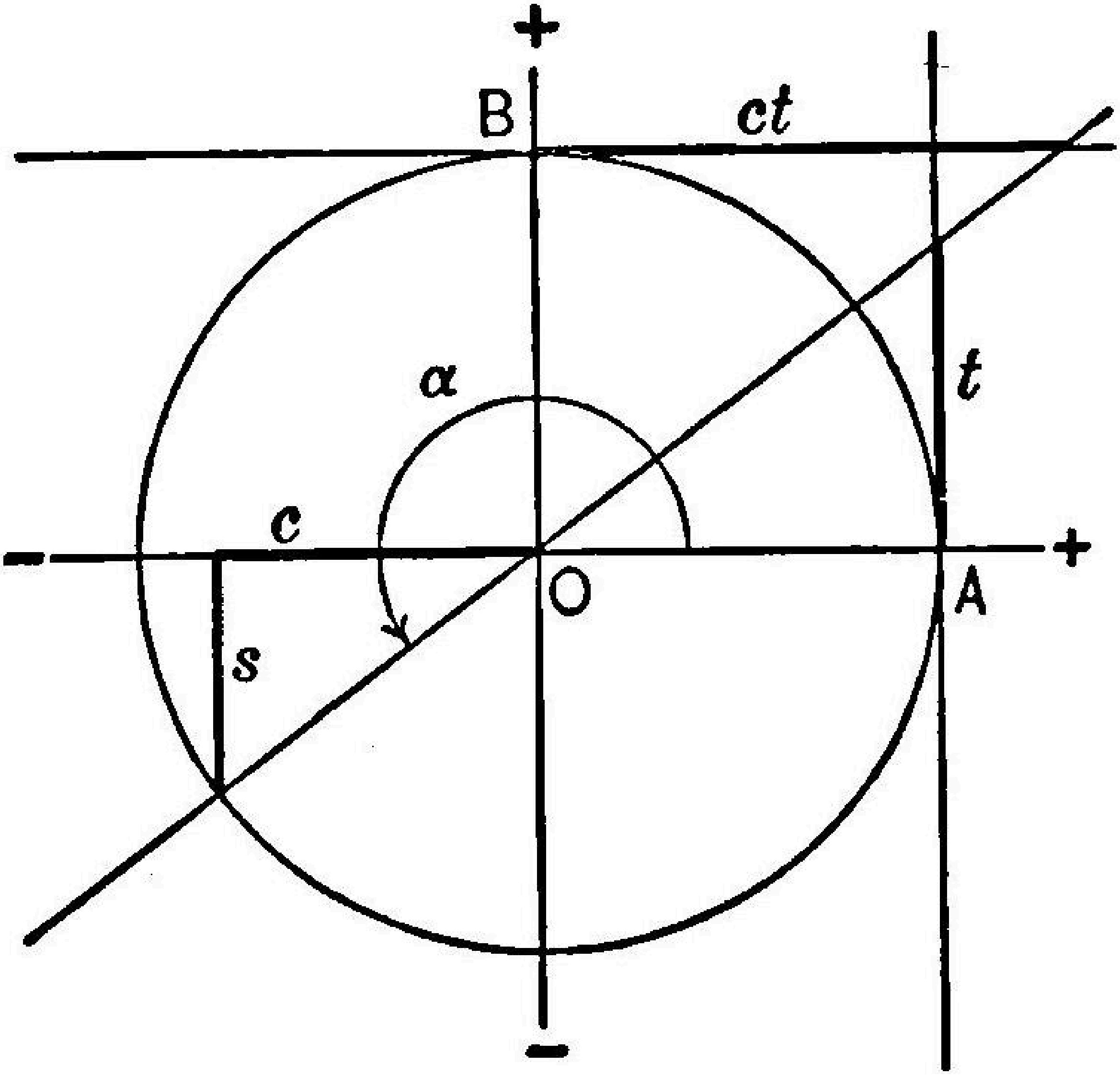

Fig. 31. Third Quadrant.

In the fourth quadrant, Fig. 32, $\sin\alpha$ is negative, as s is downward, but $\cos\alpha$ is again positive, as c is toward the right;

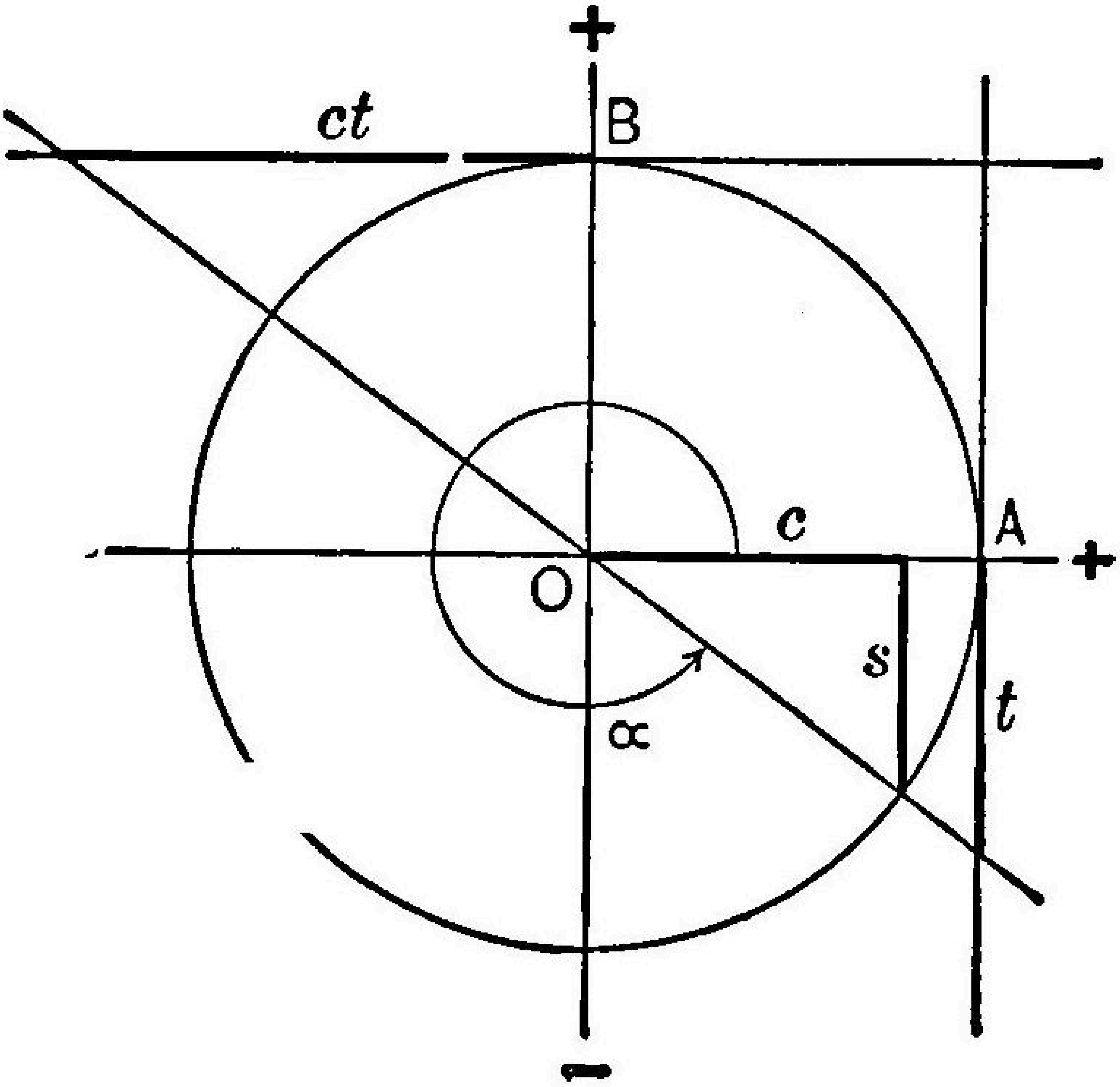

Fig. 32. Fourth Quadrant.

$\tan\alpha$ and $\cot\alpha$ are both negative, as seen from t and ct in Fig. 32.

In the fifth quadrant all the trigonometric functions again have the same values as in the first quadrant, Fig. 28 , that is, 360 deg ., or 2π, or a multiple thereof, can be added to, or subtracted from the angle α, without changing the trigonometric functions, but these functions repeat after every 360 deg ., or 2π; that is, have 2π or 360 deg. as their period.

SIGNS OF FUNCTIONS

Function.	Positive.	Negative.	
$\sin\alpha$	1st and 2d	3d and 4th quadrant	{
$\cos\alpha$	1st and 4th	2d and 3d	
$\tan\alpha$	1st and 3d	2d and 4th "	
$\cot\alpha$	1st and 3d	2d and 4th	

68. Relations between $\sin\alpha$ and $\cos\alpha$. Between $\sin\alpha$ and $\cos\alpha$ the relation,

$$\sin^2\alpha + \cos^2\alpha = 1 \tag{4}$$

exists; hence,

$$\left.\begin{aligned}\sin\alpha &= \sqrt{1-\cos^2\alpha}\\ \cos\alpha &= \sqrt{1-\sin^2\alpha}\end{aligned}\right\} \tag{4a}$$

Equation (4) is one of those which is frequently used in both directions. For instance, 1 may be substituted for the sum of the squares of $\sin\alpha$ and $\cos\alpha$, while in other cases $\sin^2\alpha+\cos^2\alpha$ may be substituted for 1 . For instance,

$$\frac{1}{\cos^2\alpha}=\frac{\sin^2\alpha+\cos^2\alpha}{\cos^2\alpha}=\left(\frac{\sin\alpha}{\cos\alpha}\right)^2+1=\tan^2\alpha+1.$$

Relations between Sines and Tangents.

$$\left.\begin{aligned}\tan\alpha &= \tfrac{\sin\alpha}{\cos\alpha};\\ \cot\alpha &= \tfrac{\cos\alpha}{\sin\alpha};\end{aligned}\right\} \tag{5}$$

hence

$$\left.\begin{aligned}\cot\alpha &= \tfrac{1}{\tan\alpha};\\ \tan\alpha &= \tfrac{1}{\cot\alpha}.\end{aligned}\right\} \ldots\ldots. \tag{5a}$$

As $\tan\alpha$ and $\cot\alpha$ are far less convenient for trigonometric calculations than $\sin\alpha$ and $\cos\alpha$, and therefore are less frequently applied in trigonometric calculations, it is not necessary to memorize the trigonometric formulas pertaining to $\tan\alpha$ and $\cot\alpha$, but where these functions occur, $\sin\alpha$ and and $\cos\alpha$ are substituted for them by equations (5), and the calculations carried out with the latter functions, and $\tan\alpha$ or $\cot\alpha$ resubstituted in the final result, if the latter contains $\frac{\sin\alpha}{\cos\alpha}$, or its reciprocal.

In electrical engineering $\tan\alpha$ or $\cot\alpha$ frequently appears as the starting-point of calculation of the phase of alternating currents. For instance, if α is the phase angle of a vector quantity, $\tan\alpha$ is given as the ratio of the vertical component over the horizontal component, or of the reactive component over the power component.

In this case, if

$$\tan\alpha=\frac{a}{b}$$
$$\sin\alpha=\frac{a}{\sqrt{a^2+b^2}},\quad\text{and}\quad\cos\alpha=\frac{b}{\sqrt{a^2+b^2}}; \tag{5b}$$

or, if

$$\cot\alpha=\frac{c}{d}$$
$$\sin\alpha=\frac{d}{\sqrt{c^2+d^2}},\quad\text{and}\quad\cos\alpha=\frac{c}{\sqrt{c^2+d^2}}.. \tag{5c}$$

The secant functions, and versed sine functions are so little used in engineering, that they are of interest only as curiosities. They are defined by the following equations:

$$\begin{gathered}\sec\alpha=\frac{1}{\cos\alpha},\\ \operatorname{cosec}\alpha=\frac{1}{\sin\alpha},\\ \sin\operatorname{vers}\alpha=1-\sin\alpha,\\ \cos\operatorname{vers}\alpha=1-\cos\alpha,\end{gathered}$$

69. Negative Angles. From the circle diagram of the trigonometric functions follows, as shown in Fig. 33, that when changing from a positive angle, that is, counterclockwise rotation, to a negative angle, that is, clockwise rotation, s, t, and ct reverse their direction, but c remains the same; that is,

$$\left.\begin{aligned}\sin(-\alpha)&=-\sin\alpha,\\ \cos(-\alpha)&=+\cos\alpha,\\ \tan(-\alpha)&=-\tan\alpha,\\ \cot(-\alpha)&=-\cot\alpha,\end{aligned}\right\}. \tag{6}$$

$\cos\alpha$ thus is an "even function," while the three others are "odd functions."

Supplementary Angles. From the circle diagram of the trigonometric functions follows, as shown in Fig. 34, that by changing from an angle to its supplementary angle, s remains in the same direction, but c, t, and ct reverse their direction, and all four quantities retain the same numerical values, thus,

$$\left.\begin{aligned}\sin(\pi-\alpha)&=+\sin\alpha,\\ \cos(\pi-\alpha)&=-\cos\alpha,\\ \tan(\pi-\alpha)&=-\tan\alpha,\\ \cot(\pi-\alpha)&=-\cot\alpha,\end{aligned}\right\}. \tag{7}$$

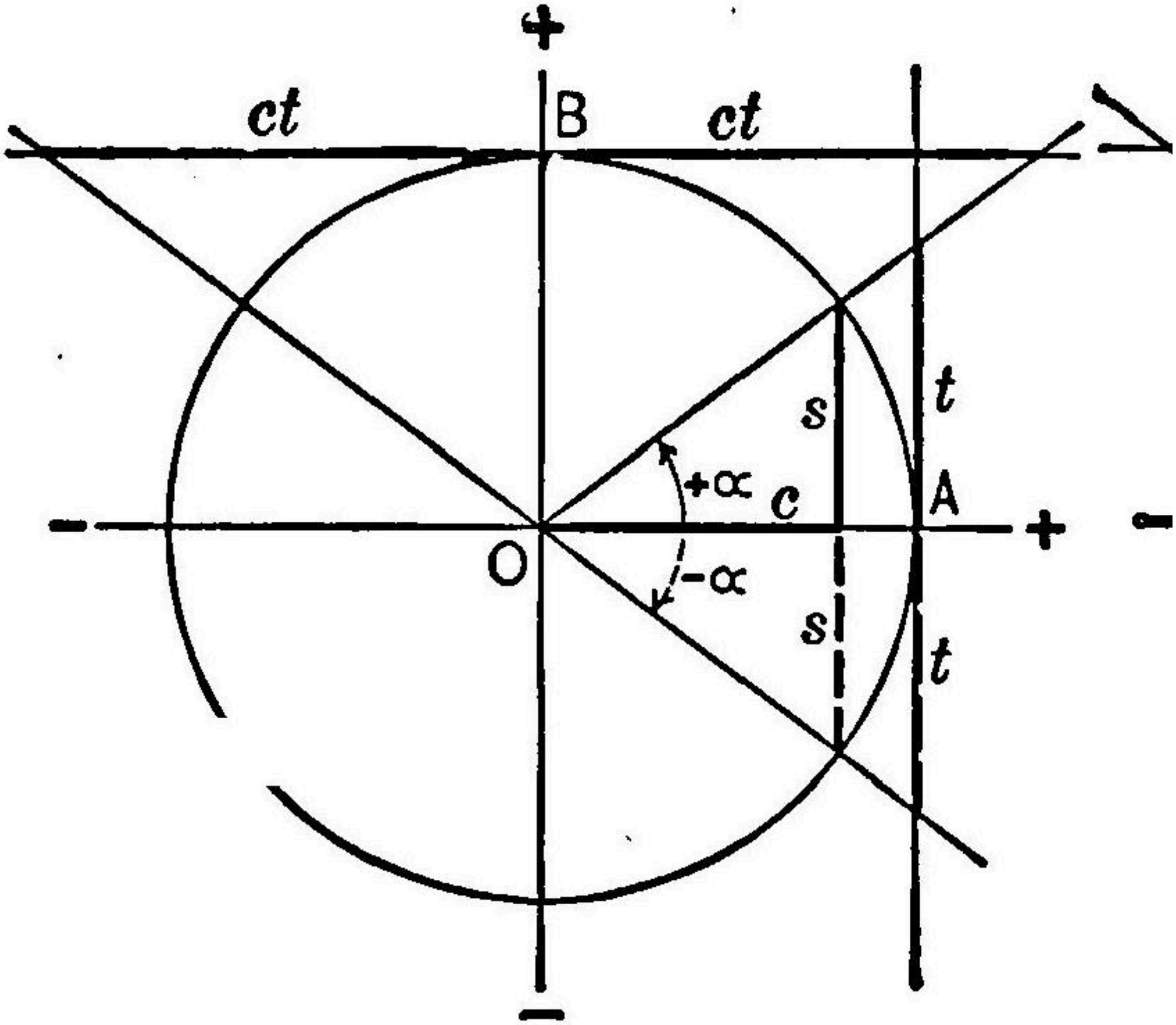

Fig. 33. Functions of Negative Angles.

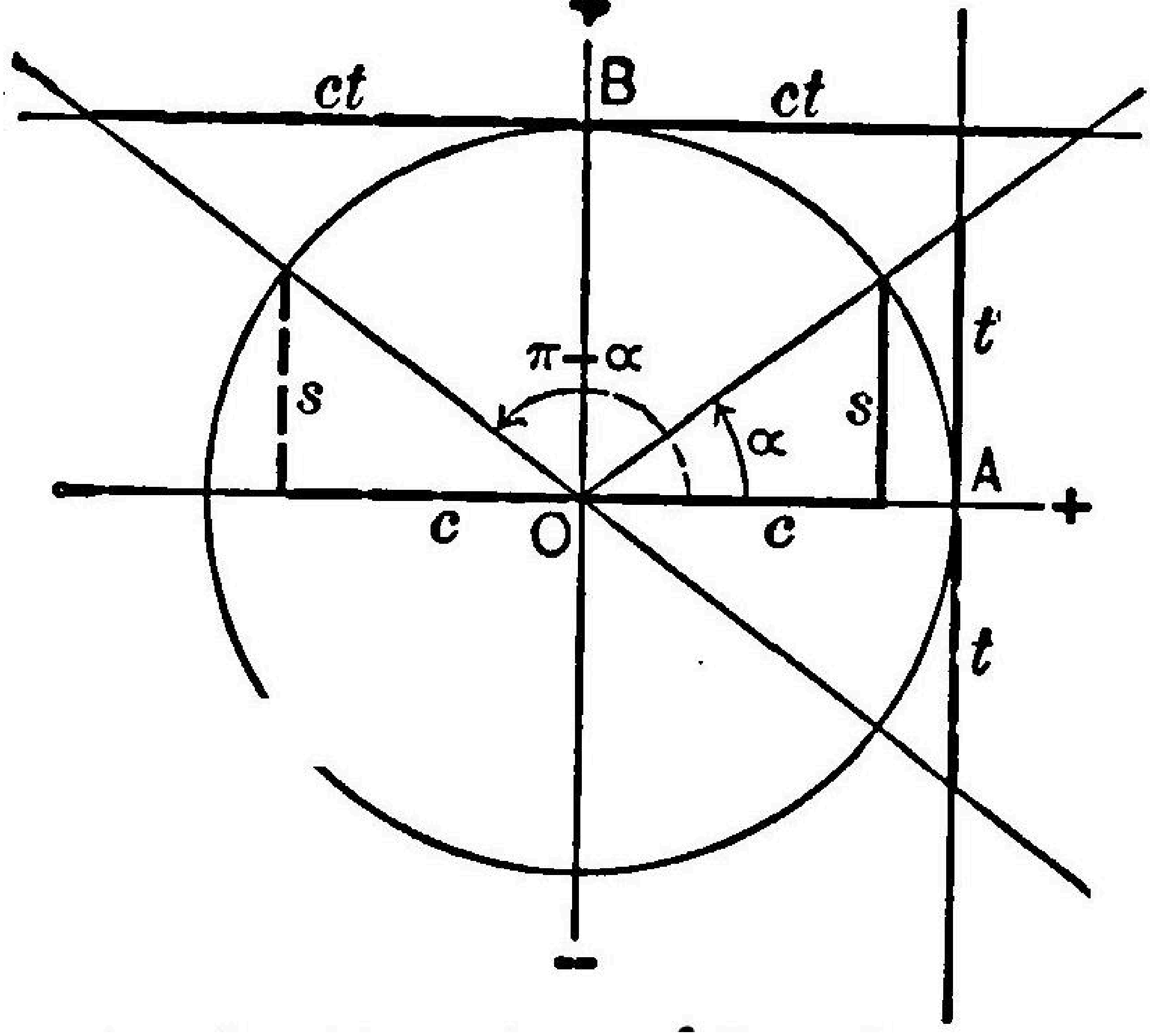

Fig. 34. Functions of Supplementary Angles.

Complementary Angles. Changing from an angle α to its complementary angle $90^\circ - \alpha$, or $\frac{\pi}{2} - \alpha$, as seen from Fig. 35, the signs remain the same, but s and c, and also t and ct exchange their numerical values, thus,

$$\left.\begin{aligned} \sin\left(\tfrac{\pi}{2}-\alpha\right) &= \cos\alpha \\ \cos\left(\tfrac{\pi}{2}-\alpha\right) &= \sin\alpha \\ \tan\left(\tfrac{\pi}{2}-\alpha\right) &= \cot\alpha \\ \cot\left(\tfrac{\pi}{2}-\alpha\right) &= \tan\alpha \end{aligned}\right\} \tag{8}$$

70. Angle $(\alpha \pm \pi)$. Adding, or subtracting π to an angle α, gives the same numerical values of the trigonometric functions

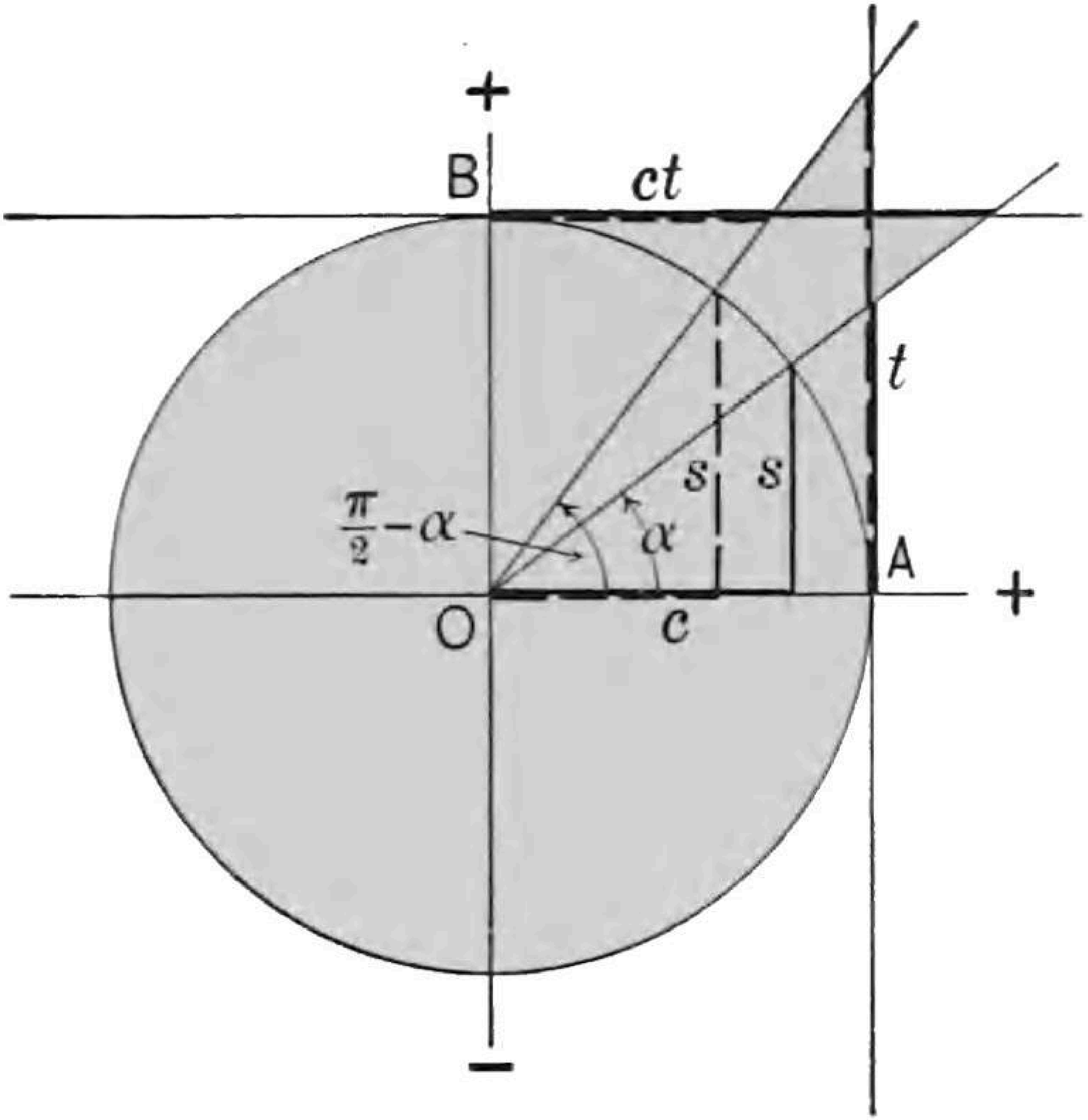

Fig. 35. Functions of Complementary Angles.

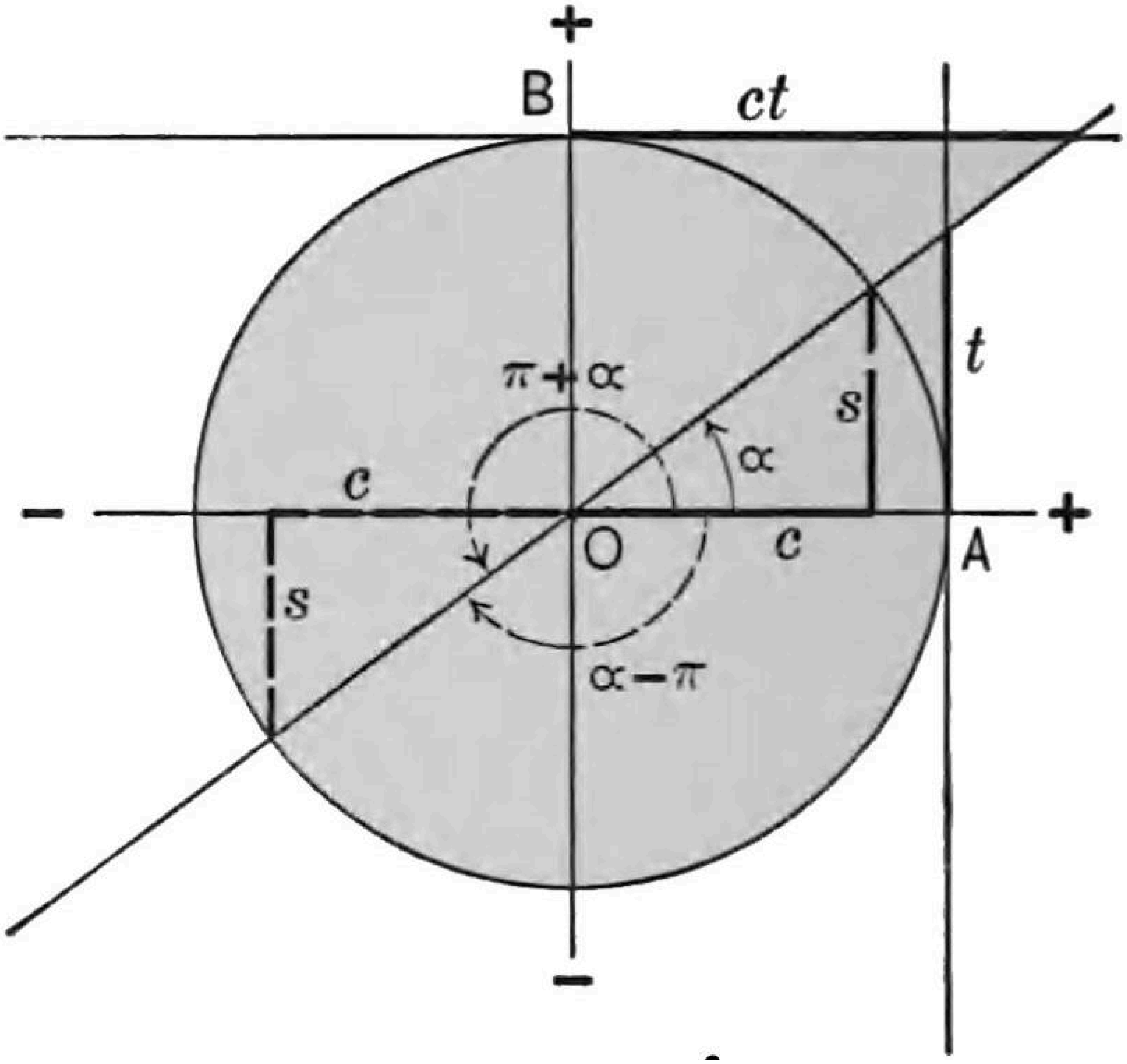

Fig. 36. Functions of Angles Plus or Minus π.

as α, as seen in Fig. 36, but the direction of s and c is reversed, while t and ct remain in the same direction, thus,

$$\left.\begin{aligned} \sin(\alpha \pm \pi) &= -\sin\alpha, \\ \cos(\alpha \pm \pi) &= -\cos\alpha, \\ \tan(\alpha \pm \pi) &= +\tan\alpha, \\ \cot(\alpha \pm \pi) &= +\cot\alpha, \end{aligned}\right\} \tag{9}$$

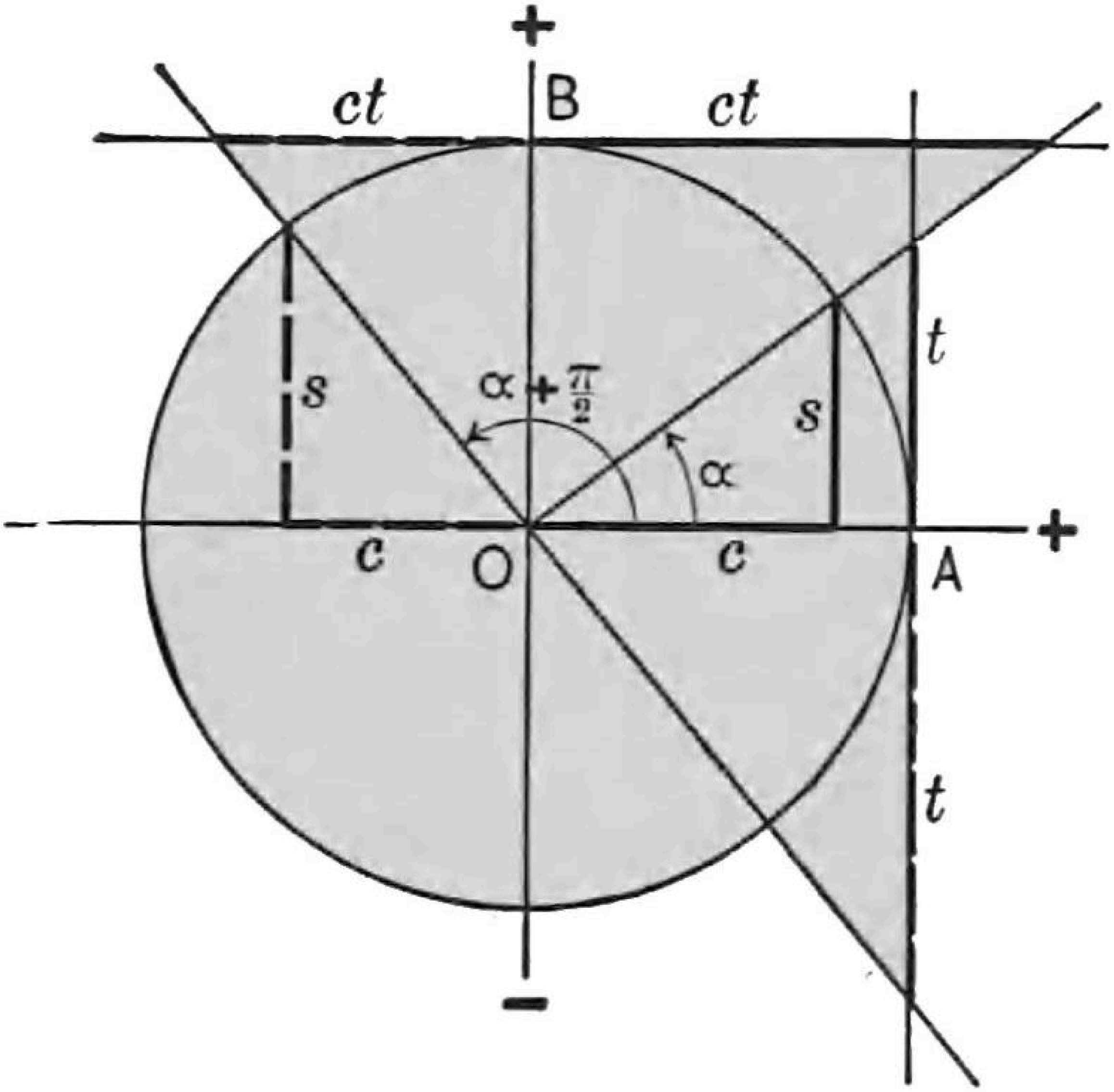

Fig. 37. Functions of Angles $+\frac{\pi}{2}$.

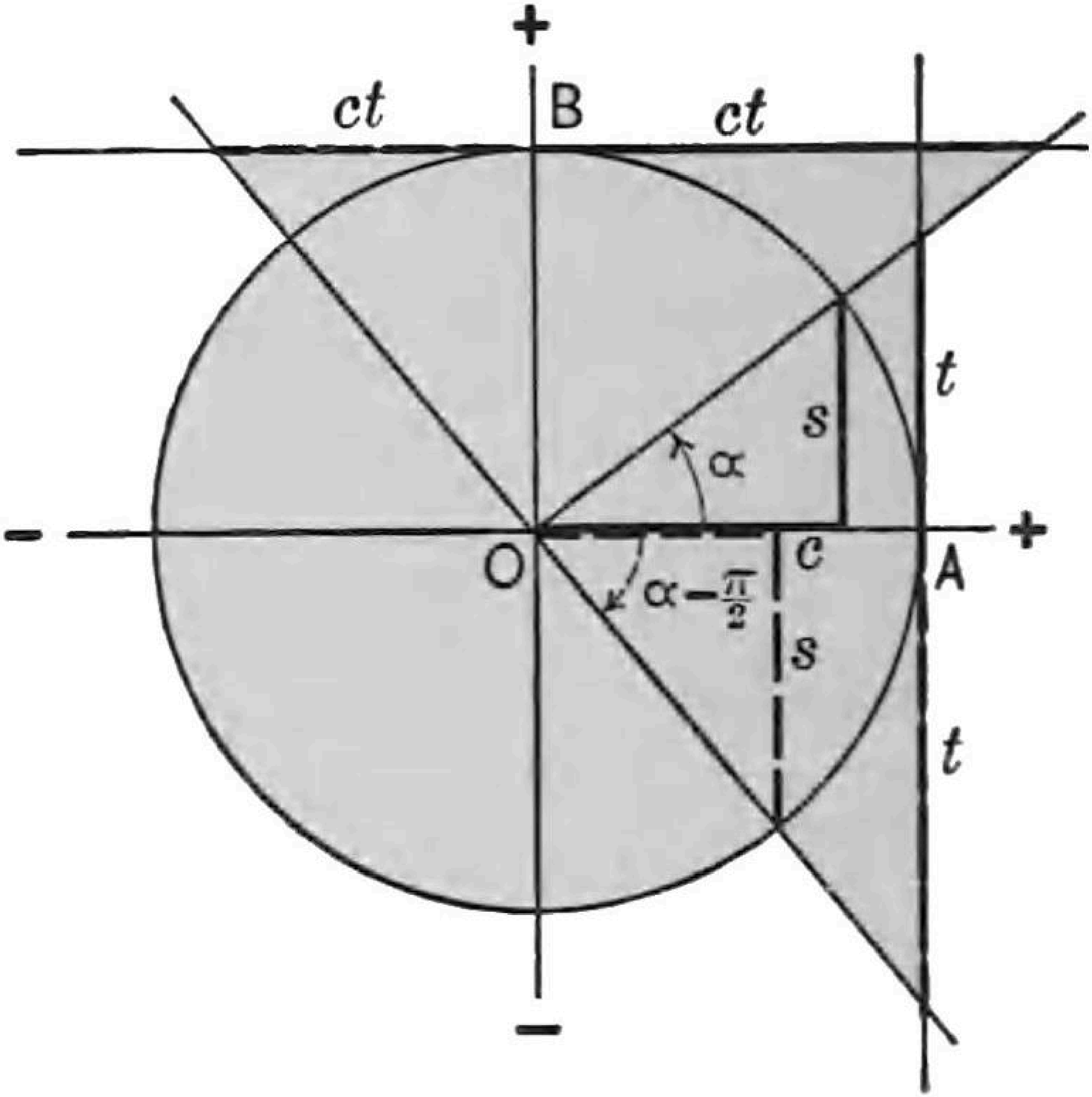

Fig. 38. Functions of Angles Minus $\frac{\pi}{2}$.

Angle $\left(\alpha \pm \frac{\pi}{2}\right)$. Adding $\frac{\pi}{2}$, or 90 deg. to an angle α, interchanges the functions, s and c, and t and ct, and also reverses the direction of the cosine, tangent, and cotangent, but leaves the sine in the same direction, since the sine is positive in the second quadrant, as seen in Fig. 37.

Subtracting $\frac{\pi}{2}$, or 90 deg. from angle α, interchanges the functions, s and c, and t and ct, and also reverses the direction, except that of the cosine, which remains in the same direction; that is, of the same sign, as the cosine is positive in the first and fourth quadrant, as seen in Fig. 38. Therefore,

$$\left.\begin{array}{l} \left.\begin{array}{l} \sin\left(\alpha+\frac{\pi}{2}\right)=+\cos\alpha \\ \cos\left(\alpha+\frac{\pi}{2}\right)=-\sin\alpha \\ \tan\left(\alpha+\frac{\pi}{2}\right)=-\cot\alpha \end{array}\right\}\ldots\ldots\ldots \\ \cot\left(\alpha+\frac{\pi}{2}\right)=-\tan\alpha, \\ \sin\left(\alpha-\frac{\pi}{2}\right)=-\cos\alpha, \\ \cos\left(\alpha-\frac{\pi}{2}\right)=+\sin\alpha, \\ \tan\left(\alpha-\frac{\pi}{2}\right)=-\cot\alpha, \\ \cot\left(\alpha-\frac{\pi}{2}\right)=-\tan\alpha. \end{array}\right\}\ldots\ldots\ldots$$

Numerical Values. From the circle diagram, Fig. 28, etc., follows the numerical values:

$$\left.\begin{array}{l|l|l|c}
\sin 0^\circ = 0 & \cos 0^\circ = 1 & \tan 0^\circ = 0 & \cot 0^\circ = \infty \\
\sin 30^\circ = \frac{1}{2} & \cos 30^\circ = \frac{1}{2}\sqrt{3} & \tan 45^\circ = 1 & \cot 45^\circ = 1 \\
\sin 45^\circ = \frac{1}{2} \vee \overline{2} & \cos 45^\circ = \frac{1}{2} \vee /\overline{2} & \tan 90^\circ = \infty & \cot 90^\circ = 0 \\
\sin 60^\circ = \frac{1}{2} \cdot \overline{3} & \cos 60^\circ = \frac{1}{2} & \tan 135^\circ = -1 & \cot 135^\circ = -1 \\
\sin 90^\circ = 1 & \cos 90^\circ = 0 & \text{etc.} & \text{etc.} \\
\sin 120^\circ = \frac{1}{2} \vee \overline{3} & & & \\
\cos 120^\circ = -\frac{1}{2} & & &
\end{array}\right\}. \tag{12}$$

71. Relations between Two Angles. The following relations are developed in text-books of trigonometry:

$$\left.\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta
\end{aligned}\right\}. \tag{13}$$

Herefrom follows, by combining these equations (13) in pairs:

$$\left.\begin{aligned}
\cos\alpha\cos\beta &= \tfrac{1}{2}\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\} \\
\sin\alpha\sin\beta &= \tfrac{1}{2}\{\cos(\alpha-\beta) - \cos(\alpha+\beta\}) \\
\sin\alpha\cos\beta &= \tfrac{1}{2}\{\sin(\alpha+\beta) + \sin(\alpha-\beta)\} \\
\cos\alpha\sin\beta &= \tfrac{1}{2}\{\sin(\alpha+\beta) - \sin(\alpha-\beta)\}
\end{aligned}\right\} \tag{14}$$

By substituting α_1 for $(\alpha+\beta)$, and β_1 for $(\alpha-\beta)$ in these equations (14), gives the equations,

$$\left.\begin{aligned}
\sin\alpha_1 + \sin\beta_1 &= 2\sin\tfrac{\alpha_1+\beta_1}{2}\cos\tfrac{\alpha_1-\beta_1}{2}, \\
\sin\alpha_1 - \sin\beta_1 &= 2\sin\tfrac{\alpha_1-\beta_1}{2}\cos\tfrac{\alpha_1+\beta_1}{2}, \\
\cos\alpha_1 + \cos\beta_1 &= 2\cos\tfrac{\alpha_1+\beta_1}{2}\cos\tfrac{\alpha_1-\beta_1}{2}, \\
\cos\alpha_1 - \cos\beta_1 &= -2\sin\tfrac{\alpha_1+\beta_1}{2}\sin\tfrac{\alpha_1-\beta_1}{2}.
\end{aligned}\right\} \tag{15}$$

These three sets of equations are the most important trigonometric formulas. Their memorizing can be facilitated by noting that cosine functions lead to products of equal functions, sine functions to products of unequal functions, and inversely, products of equal functions resolve into cosine, products of unequal functions into sine functions. Also cosine functions show a reversal of the sign, thus: the cosine of a sum is given by a difference of products, the cosine of a difference by a sum, for the reason that with increasing angle the cosine function decreases, and the cosine of a sum of angles thus would be less than the cosine of the single angle.

Double Angles. From (13) follows, by substituting α for β :

$$\left.\begin{aligned}
\sin 2\alpha &= 2\sin\alpha\cos\alpha, \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha, \\
&= 2\cos^2\alpha - 1, \\
&= 1 - 2\sin^2\alpha.
\end{aligned}\right\} \tag{16}$$

Herefrom follow

$$\left.\sin^2\alpha = \frac{1-\cos 2\alpha}{2} \quad \text{and} \quad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}\right\}. \tag{16a}$$

72. Differentiation.

$$\left.\begin{aligned}
\tfrac{d}{d\alpha}(\sin\alpha) &= +\cos\alpha, \\
\tfrac{d}{d\alpha}(\cos\alpha) &= -\sin\alpha.
\end{aligned}\right\}\ldots\cdot\ldots \tag{17}$$

The sign of the latter differential is negative, as with an increase of angle α, the $\cos\alpha$ decreases.

Integration.

$$\left.\begin{array}{l}\int \sin\alpha d\alpha = -\cos\alpha \\ \int \cos\alpha d\alpha = +\sin\alpha\end{array}\right\}\ldots\ldots\ldots \tag{18}$$

Herefrom follow the definite integrals:

$$\int_c^{c+2\pi} \sin(\alpha + a)d\alpha = 0; \tag{18a}$$

$$\int_c^{c+2\pi} \cos(\alpha + a)d\alpha = 0; \tag{18b}$$

$$\int_c^{c+\pi} \sin(\alpha + a)d\alpha = 2\cos(c + a)$$

$$\int_c^{c+\pi} \cos(\alpha + a)d\alpha = -2\sin(c + a);$$

$$\left.\begin{array}{l}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sin\alpha d\alpha = 0 \\ \int_0^{\pi} \cos\alpha d\alpha = 0\end{array}\right\}. \tag{18c}$$

$$\left.\begin{array}{l}\int_0^{\frac{\pi}{2}} \sin\alpha d\alpha = +1; \\ \int_0^{\frac{\pi}{2}} \cos\alpha d\alpha = +1.\end{array}\right\}\ldots\ldots\ldots\ldots \tag{18d}$$

73. Binomial. One of the most frequent trigonometric operations in electrical engineering is the transformation of the binomial, $a\cos\alpha + b\sin\alpha$, into a single trigonometric function, by the substitution, $a = c\cos p$ and $b = c\sin p$; hence,

$$a\cos\alpha + b\sin\alpha = c\cos(\alpha - p), \tag{19}$$

where

$$c = \sqrt{a^2 + b^2} \quad \text{and} \quad \tan p = \frac{b}{a}; \quad \ldots \quad . \tag{20}$$

or, by the transformation, $a = c\sin q$ and $b = c\cos q$,

$$a\cos\alpha + b\sin\alpha = c\sin(\alpha + q), \tag{21}$$

where

$$c = \sqrt{a^2 + b^2} \quad \text{and} \quad \tan q = \frac{a}{b}\ldots \quad \ldots \tag{22}$$

74. Polyphase Relations.

$$\left.\begin{array}{l}\sum_1^n i\cos\left(\alpha + a \pm \frac{2mi\pi}{n}\right) = 0 \\ \sum_1^n i\sin\left(\alpha + a \pm \frac{2mi\pi}{n}\right) = 0\end{array}\right\}. \tag{23}$$

where m and n are integer numbers.

Proof. The points on the circle which defines the trigonometric function, by Fig. 28, of the angles $\left(\alpha + a \pm \frac{2mi\pi}{n}\right)$. are corners of a regular polygon, inscribed in the circle and therefore having the center of the circle as center of gravity. For instance, for $n = 7, m = 2$, they are shown as $P_1, P_2, \ldots P_7$, in Fig. 39. The cosines of these angles are the projections on the vertical, the sines, the projections on the horizontal diameter, and as the sum of the projections of the corners of any polygon,

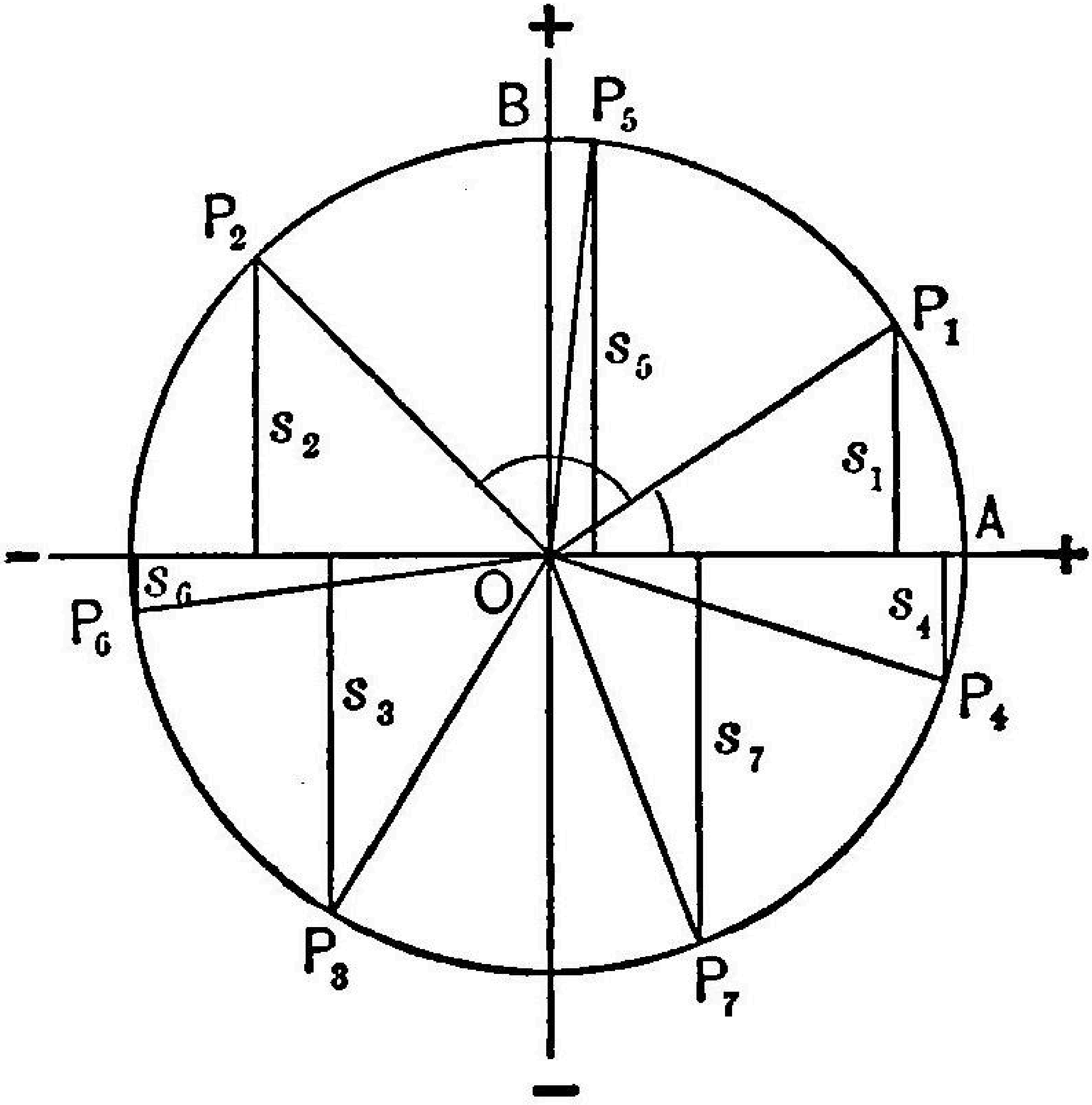

Fig. 39. Polyphase Relations.

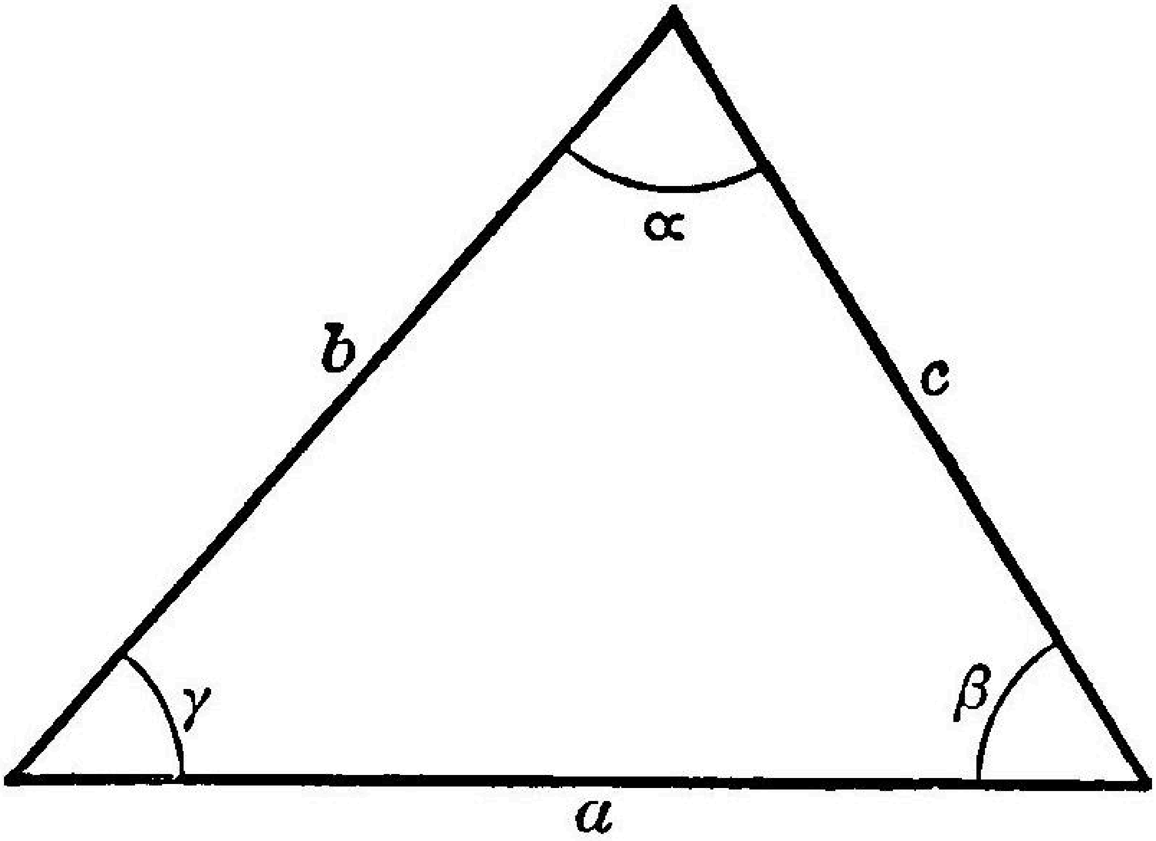

Fig. 40. Triangle.

on any line going through its center of gravity, is zero, both sums of equation (23) are zero.

$$\left.\begin{array}{l}\sum_1^n i\cos\left(\alpha+a\pm\frac{2mi\pi}{n}\right)\cos\left(\alpha+b\pm\frac{2mi\pi}{n}\right)=\frac{n}{2}\cos(a--b),\\ \sum_1^n i\sin\left(\alpha+a\pm\frac{2mi\pi}{n}\right)\sin\left(\alpha+b\pm\frac{2mi\pi}{n}\right)=\frac{n}{2}\cos(a-b),\\ \sum_1^n i\sin\left(\alpha+a\pm\frac{2mi\pi}{n}\right)\cos\left(\alpha+b\pm\frac{2mi\pi}{n}\right)=\frac{n}{2}\sin(a-b).\end{array}\right\} \tag{24}$$

These equations are proven by substituting for the products the single functions by equations (14), and substituting them in equations (23).
75. Triangle. If in a triangle α, β, and γ are the angles, opposite respectively to the sides a, b, c, Fig. 40, then,

$$\sin\alpha\div\sin\beta\div\sin\gamma=a\div b\div c,\ldots\quad. \tag{25}$$

or

$$\left.\begin{array}{l}\cos\gamma=\frac{a^2+b^2-c^2}{2ab};\\ c^2=a^2+b^2-2ab\cos\gamma.\end{array}\right\}\left.\begin{array}{ll}\text{Area} & =\frac{ab\sin\gamma}{2}\\ & =\frac{c^2\sin\alpha\sin\beta}{2\sin\gamma}\end{array}\right\}. \tag{27}$$

B. TRIGONOMETRIC SERIES.

76. Engineering phenomena usually are either constant, transient, or periodic. Constant, for instance, is the terminal voltage of a storage-battery and the current taken from it through a constant resistance. Transient phenomena occur during a change in the condition of an electric circuit, as a change of load; or, disturbances entering the circuit from the outside or originating in it, etc. Periodic phenomena are the alternating currents and voltages, pulsating currents as those produced by rectifiers, the distribution of the magnetic flux in the air-gap of a machine, or the distribution of voltage around the commutator of the direct-current machine, the motion of the piston in the steam-engine cylinder, the variation of the mean daily temperature with the seasons of the year, etc.

The characteristic of a periodic function, $y=f(x)$, is, that at constant intervals of the independent variable x, called cycles or periods, the same values of the dependent variable y occur.

Most periodic functions of engineering are functions of time or of space, and as such have the characteristic of univalence; that is, to any value of the independent variable x can correspond only one value of the dependent variable y. In other words, at any given time and given point of space, any physical phenomenon can have one numerical value only, and therefore must be represented by a univalent function of time and space.

Any univalent periodic function,

$$y = f(x), \tag{1}$$

can be expressed by an infinite trigonometric series, or Fourier series, of the form,

$$\begin{aligned} y = a_0 + a_1 \cos cx + a_2 \cos 2cx + a_3 \cos 3cx + \dots \\ + b_1 \sin cx + b_2 \sin 2cx + b_3 \sin 3cx + \dots \end{aligned} \tag{2}$$

or, substituting for convenience, $cx = \theta$, this gives

$$\begin{aligned} y = a_0 + a_1 \cos \theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \dots \\ + b_1 \sin \theta + b_2 \sin 2\theta + b_3 \sin 3\theta + \dots \end{aligned} \tag{3}$$

or, combining the sine and cosine functions by the binomial (par. 73),

$$\left. \begin{aligned} y &= a_0 + c_1 \cos(\theta - \beta_1) + c_2 \cos(2\theta - \beta_2) + c_3 \cos(3\theta - \beta_3) + \dots \\ &= a_0 + c_1 \sin(\theta + \gamma_1) + c_2 \sin(2\theta + \gamma_2) + c_3 \sin(3\theta + \gamma_3) + \dots \end{aligned} \right\}, \tag{4}$$

where
or

$$\left. \begin{aligned} c_n &= \sqrt{{a_n}^2 + {b_n}^2}; \\ \tan \beta_n &= \tfrac{b_n}{a_n}; \\ \tan \gamma_n &= \tfrac{a_n}{b_n}. \end{aligned} \right\} \tag{5}$$

$$\left. \begin{array}{l} \text{is given by showing that the coeffic} \\ \text{s (3) can be determined from the nume} \\ \text{c function (1), thus,} \\ y = f(x) = f_0(\theta) \end{array} \right\} \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

values of the periodic function (1), thus,

Since, however, the trigonometric function, and therefore also the series of trigonometric functions (3) is univalent, it follows that the periodic function (6), $y = f_0(\theta)$, must be univalent, to be represented by a trigonometric series.
77. The most important periodic functions in electrical engineering are the alternating currents and e.m.fs. Usually they are, in first approximation, represented by a single trigonometric function, as:

$$i = i_0 \cos(\theta - \omega);$$

or,

$$e = e_0 \sin(\theta - \delta);$$

that is, they are assumed as sine waves.

Theoretically, obviously this condition can never be perfectly attained, and frequently the deviation from sine shape is sufficient to require practical consideration, especially in those cases, where the electric circuit contains electrostatic capacity, as is for instance, the case with long-distance transmission lines, underground cable systems, high potential transformers, etc.

However, no matter how much the alternating or other periodic wave differs from simple sine shape-that is, however much the wave is "distorted," it can always be represented by the trigonometric series (3).

As illustration the following applications of the trigonometric series to engineering problems may be considered:
(A) The determination of the equation of the periodic function; that is, the evolution of the constants a_n and b_n of the trigonometric series, if the numerical values of the periodic function are given. Thus, for instance, the wave of an alternator may be taken by oscillograph or wave-meter, and by measuring from the oscillograph, the numerical values of the periodic function are derived for every 10 degrees, or every 5 degrees, or every degree, depending on the accuracy required. The problem then is, from the numerical values of the wave, to determine its equation. While the oscillograph shows the shape of the wave, it obviously is not possible therefrom to

calculate other quantities, as from the voltage the current under given circuit conditions, if the wave shape is not first represented by a mathematical expression. It therefore is of importance in engineering to translate the picture or the table of numerical values of a periodic function into a mathematical expression thereof.

(B) If one of the engineering quantities, as the e.m.f. of an alternator or the magnetic flux in the air-gap of an electric machine, is given as a general periodic function in the form of a trigonometric series, to determine therefrom other engineering quantities, as the current, the generated e.m.f., etc.

A. Evaluation of the Constants of the Trigonometric Series from the Instantaneous Values of the Periodic Function.

78. Assuming that the numerical values of a univalent periodic function $y=f_0(\theta)$ are given; that is, for every value of θ, the corresponding value of y is known, either by graphical representation, Fig. 41; or, in tabulated form, Table I, but
the equation of the periodic function is not known. It can be represented in the form,

$$\begin{aligned} y = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \ldots + a_n \cos n\theta + \ldots \\ + b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta - \ldots + b_n \sin n\theta + \ldots, \end{aligned} \tag{7}$$

and the problem now is, to determine the coefficients $a_0, a_1, a_2 \ldots b_1, b_2 \ldots$

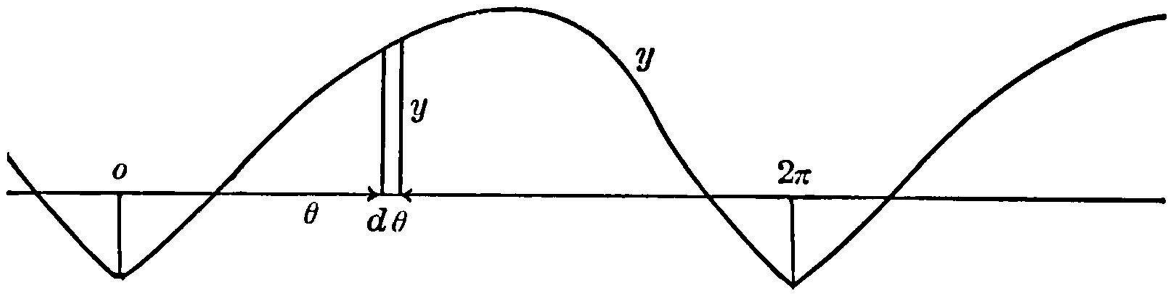

Fig. 41. Periodic Functions.

TABLE I.

θ	y	θ	y	θ	y	θ	ν
0	-60	90	+ 50	180	+122	270	+85
10	-49	100	+ 61	190	+124	280	+65
20	-38	110	+ 71	200	+126	290	+35
30	-26	120	+ 81	210	+125	300	+17
40	-12	130	+ 90	220	+123	310	0
50	0	140	+ 99	230	+120	320	-13
60	+11	150	+107	240	+116	330	-26
70	+27	160	+114	250	+110	340	-38
80	+39	170	+119	260	+100	350	-49
90	+50	180	+122	270	+ 85	360	-60

Integrate the equation (7) between the limits 0 and 2π :

$$\begin{aligned} \int_0^{2\pi} y d\theta = a_0 \int_0^{2\pi} d\theta + a_1 \int_0^{2\pi} \cos\theta d\theta + a_2 \int_0^{2\pi} \cos 2\theta d\theta + \ldots \\ + a_n \int_0^{2\pi} \cos n\theta d\theta + \ldots + b_1 \int_0^{2\pi} \sin\theta d\theta + \\ + b_2 \int_0^{2\pi} \sin 2\theta d\theta + \ldots + b_n \int_0^{2\pi} \sin n\theta d\theta + \ldots \\ = a_0/\theta/_0^{2\pi} + a_1/\sin\theta/_0^{2\pi} + a_2/\frac{\sin 2\theta}{2}/_0^{2\pi} + \ldots \\ + a_n/\frac{\sin n\theta}{n}/_0^{2\pi} + \ldots - b_1/\cos\theta/_0^{2\pi} \\ - b_2/\frac{\cos 2\theta}{2}/_0^{2\pi} - \ldots - b_n/\frac{\cos n\theta}{n}/_0^{2\pi} + \ldots \end{aligned}$$

All the integrals containing trigonometric functions vanish, as the trigonometric function has the same value at the upper limit 2π as at the lower limit 0 , that is,

$$/\frac{\cos n\theta}{n}/_0^{2\pi} = \frac{1}{n}(\cos 2n\pi - \cos 0) = 0$$
$$/\frac{\sin n\theta}{n}/_0^{2\pi} = \frac{1}{n}(\sin 2n\pi - \sin 0) = 0$$

and the result is

$$\int_0^{2\pi} yd\theta = a_0/0/_0^{2\pi} = 2\pi a_0$$

hence

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} yd\theta \tag{8}$$

$yd\theta$ is an element of the area of the curve y, Fig. 41 , and $\int_0^{2\pi} yd\theta$ thus is the area of the periodic function y, for one period; that is,

$$a_0 = \frac{1}{2\pi}A, \tag{9}$$

where $A =$ area of the periodic function $y = f_0(\theta)$, for one period; that is, from $\theta = 0$ to $\theta = 2\pi$.
2π is the horizontal width of this area A, and $\frac{A}{2\pi}$ thus is the area divided by the width of it; that is, it is the average height of the area A of the periodic function y; or, in other words, it is the average value of y. Therefore,

$$a_0 = \text{avg}\cdot(y)_0^{2\pi}\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \tag{10}$$

The first coefficient, a_0, thus, is the average value of the instantaneous values of the periodic function y, between $\theta = 0$ and $\theta = 2\pi$.

Therefore, averaging the values of y in Table I, gives the first constant a_0.
79. To determine the coefficient a_n, multiply equation (7) by $\cos n\theta$, and then integrate from 0 to 2π, for the purpose of making the trigonometric functions vanish. This gives

$$\begin{aligned}\int_0^{2\pi} y\cos n\theta d\theta = a_0\int_0^{2\pi}\cos n\theta d\theta + a_1\int_0^{2\pi}\cos n\theta\cos\theta d\theta +\\ + a_2\int_0^{2\pi}\cos n\theta\cos 2\theta d\theta + \ldots + a_n\int_0^{2\pi}\cos^2 n\theta d\theta + \ldots\\ + b_1\int_0^{2\pi}\cos n\theta\sin\theta d\theta + b_2\int_0^{2\pi}\cos n\theta\sin 2\theta d\theta + \ldots\\ + b_n\int_0^{2\pi}\cos n\theta\sin n\theta d\theta + \ldots\end{aligned}$$

Hence, by the trigonometric equations of the preceding section:

$$\begin{aligned}\int_0^{2\pi} y\cos n\theta d\theta = a_0\int_0^{2\pi}\cos n0 d\theta + a_1\int_0^{2\pi}\frac{1}{2}[\cos(n+1)\theta + \cos(n-1)\theta]d\theta\\ + a_2\int_0^{2\pi}\frac{1}{2}[\cos(n+2)\theta + \cos(n-2)\theta]d\theta + \ldots\\ + a_n\int_0^{2\pi}\frac{1}{2}(1 + \cos 2n\theta)d\theta + \ldots\\ + b_1\int_0^{2\pi}\frac{1}{2}[\sin(n+1)\theta - \sin(n-1)\theta]d\theta\\ + b_2\int_0^{2\pi}\frac{1}{2}[\sin(n+2)\theta - \sin(n-2)\theta]d\theta + \ldots\\ + b_n\int_0^{2\pi}\frac{1}{2}\sin 2n\theta d\theta + \ldots\end{aligned}$$

All these integrals of trigonometric functions give trigonometric functions, and therefore vanish between the limits 0 and 2π, and there only remains the first term of the integral multiplied with a_n, which does not contain a trigonometric function, and thus remains finite:

$$a_n \int_0^{2\pi} \frac{1}{2} d\theta = a_n \left(\frac{\theta}{2}\right)_0^{2\pi} = a_n \pi$$

and therefore,

$$\int_0^{2\pi} y \cos n\theta d\theta = a_n \pi$$

hence

$$a_n = \frac{1}{\pi} \int_0^{2\pi} y \cos n\theta d\theta \ldots\ldots\ldots \tag{11}$$

If the instantaneous values of y are multiplied with $\cos n\theta$, and the product $y_n = y \cos n\theta$ plotted as a curve, $y \cos n\theta d\theta$ is an element of the area of this curve, shown for $n = 3$ in Fig. 42, and thus $\int_0^{2\pi} y \cos n0 d\theta$ is the area of this curve; that is,

$$a_n = \frac{1}{\pi} A_n, \quad . \quad . \quad . \quad . \tag{12}$$

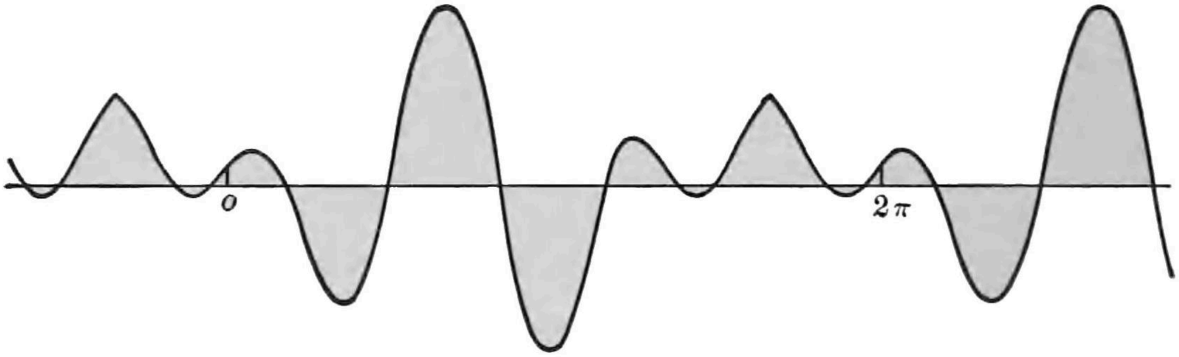

Fig. 42. Curve of $y \cos 30$.

where A_n is the area of the curve $y \cos n\theta$, between $\theta = 0$ and $\theta = 2\pi$.

As 2π is the width of this area A_n, $\frac{A_n}{2\pi}$ is the average height of this area; that is, is the average value of $y \cos n\theta$, and $\frac{1}{\pi} A_n$ thus is twice the average value of $y \cos n\theta$; that is,

$$a_n = 2 \,\text{avg.}\, (y \cos n\theta)_0^{2\pi}. \tag{13}$$

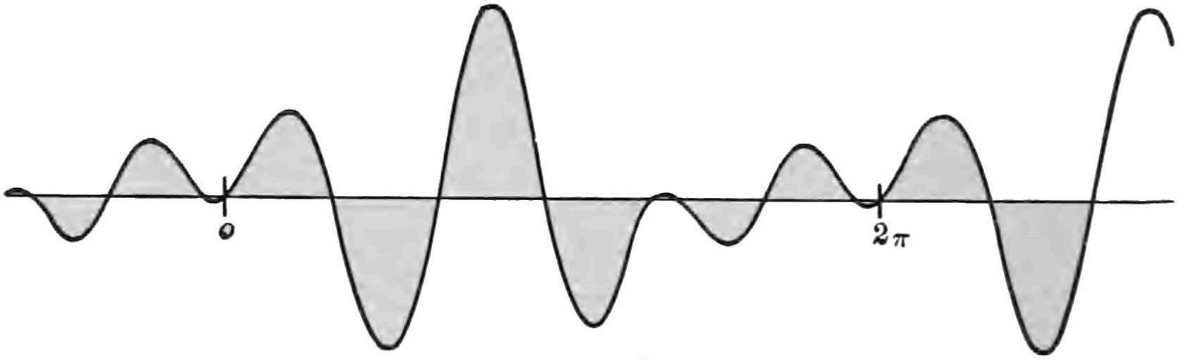

Fig. 43. Curve of $y \sin 3\theta$.

The coefficient a_n of $\cos n\theta$ is derived by multiplying all the instantaneous values of y by $\cos n\theta$, and taking twice the average of the instantaneous values of this product $y \cos n\theta$.
80. b_n is determined in the analogous manner by multiplying y by $\sin n\theta$ and integrating from 0 to 2π; by the area of the curve $y \sin n\theta$, shown in Fig. 43, for $n = 3$,

$$\begin{aligned}
&\int_0^{2\pi} y\sin n\theta d\theta = a_0\int_0^{2\pi}\sin n\theta d\theta + a_1\int_0^{2\pi}\sin n\theta\cos\theta d\theta\\
&+a_2\int_0^{2\pi}\sin n\theta\cos 2\theta d\theta + \ldots + a_n\int_0^{2\pi}\sin n\theta\cos n\theta d\theta + \ldots\\
&+b_1\int_0^{2\pi}\sin n\theta\sin\theta d\theta + b_2\int_0^{2\pi}\sin n\theta\sin 2\theta d\theta + \ldots\\
&+b_n\int_0^{2\pi}\sin^2 n\theta d\theta + \ldots\\
&= a_0\int_0^{2\pi}\sin n\theta d\theta + a_1\int_0^{2\pi}\frac{1}{2}[\sin(n-1)\theta + \sin(n-1)\theta]d\theta\\
&+a_2\int_0^{2\pi}\frac{1}{2}[\sin(n+2)\theta + \sin(n-2)\theta]d\theta + \ldots\\
&+a_n\int_0^{2\pi}\frac{1}{2}\sin 2n\theta l\theta + \ldots\\
&+b_1\int_0^{2\pi}\frac{1}{2}[\cos(n-1)\theta - \cos(n+1)\theta]d\theta\\
&+b_2\int_0^{2\pi}\frac{1}{2}[\cos(n-2)\theta - \cos(n+2)\theta]d\theta + \ldots\\
&+b_n\int_0^{2\pi}\frac{1}{2}[1 - \cos 2n\theta]d\theta + \ldots\\
&= b_n\int_0^{2\pi}\frac{1}{2}d\theta = b_n\pi;
\end{aligned}$$

hence,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} y\sin n\theta d0\ldots\ldots &(14)\\
&= \frac{1}{\pi}{A_n}',\ldots\ldots\ldots &(15)
\end{aligned}$$

where ${A_n}'$ is the area of the curve ${y_n}' = y\sin n\theta$. Hence,

$$b_n = 2\,\text{avg.}\,(y\sin n\theta)_0^{2\pi} \qquad (16)$$

and the coefficient of $\sin n\theta$ thus is derived by multiplying the instantaneous values of y with $\sin n\theta$, and then averaging, as twice the average of $y\sin n\theta$.

81. Any univalent periodic function, of which the numerical values y are known, can thus be expressed numerically by the equation,

$$\begin{aligned}
y = a_0 + a_1\cos\theta + a_2\cos 2\theta + \ldots + a_n\cos n\theta + \ldots&\\
+ b_1\sin\theta + b_2\sin 2\theta + \ldots + b_n\sin n\theta + \ldots,& \qquad (17)
\end{aligned}$$

where the coefficients $a_0, a_1, a_2, \ldots b_1, b_2 \ldots$, are calculated as the averages:

$$\left.\begin{array}{ll}
a_0 = \text{ avg. }(y)_0{}^{2\pi}; & \\
a_1 = 2\text{ avg. }(y\cos\theta)_0{}^{2\pi}; & b_1 = 2\text{ avg. }(y\sin\theta)_0{}^{2\pi};\\
a_2 = 2\text{ avg. }(y\cos 2\theta)_0{}^{2\pi}; & b_2 = 2\text{ avg. }(y\sin 2\theta)_0{}^{2\pi};\\
a_n = 2\text{ avg. }(y\cos n\theta)_0{}^{2\pi}; & b_n = 2\text{ avg. }(y\sin n\theta)_0{}^{2\pi};
\end{array}\right\}. \qquad (18)$$

Hereby any individual harmonic can be calculated, without calculating the preceding harmonics.

For instance, let the generator e.m.f. wave, Fig. 44, Table II, column 2, be impressed upon an underground cable system

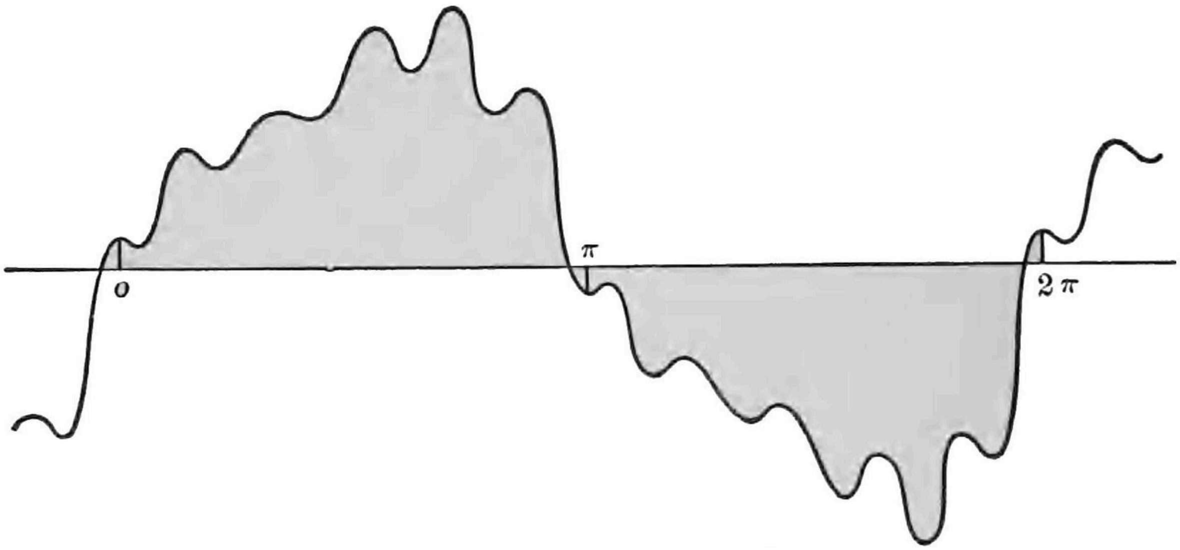

Fig. 44. Generator e.m.f. wave.

of such constants (capacity and inductance), that the natural frequency of the system is 670 cycles per second, while the generator frequency is 60 cycles. The natural frequency of the circuit is then close to that of the 11 th harmonic of the generator wave, 660 cycles, and if the generator voltage contains an appreciable 11th harmonic, trouble may result from a resonance rise of voltage of this frequency; therefore, the 11th harmonic of the generator wave is to be determined, that is, a_{11} and b_{11} calculated, but the other harmonics are of less importance.

Table II

θ	y	$\cos 11\theta$	$\sin 11\theta$	$y \cos 11\theta$	$y \sin 11\theta$
0	5	+1.000	0	+5.0	0
10	4	-0.342	+0.940	-1.4	+ 3.8
20	20	-0.766	-0.643	-15.3	-12.9
30	22	+0.866	-0.500	+19.1	-11.0
40	19	+0.174	+0.985	+3.3	+18.7
50	25	-0.985	-0.174	-24.6	- 4.3
60	29	+0.500	-0.866	+14.5	-25.1
70	29	+0.643	+0.766	+18.6	+22.2
80	30	-0.940	+0.342	-28.2	+10.3
90	38	0	-1.000	0	-38.0
100	46	+0.940	+0.342	+43.3	+15.7
110	38	-0.643	+0.766	-24.4	+29.2
120	41	-0.500	-0.866	-20.5	-35.5
130	50	+0.985	-0.174	+49.2	- 8.7
140	32	-0.174	+0.985	-5.6	+31.5
150	30	-0.866	-0.500	-26.0	
160	33	+0.766	-0.643	+25.3	-15.0
170	7	+0.342	+0.940	+2.2	-21.3
180	-5				
		Total Divided by 9. $\left(2/m^{\frac{1}{2}}4\right)$		$+34.5$ $+3.83 = a_{11}$	-29.8 $-3.31 = \mathrm{b}_{11}$

In the third column of Table II thus are given the values of $\cos 11\theta$, in the fourth column $\sin 11\theta$, in the fifth column $y \cos 11\theta$, and in the sixth column $y \sin 11\theta$. The former gives as average +1.915 , hence $a_{11} = +3.83$, and the latter gives as average -1.655 , hence $b_{11} = -3.31$, and the 11th harmonic of the generator wave is

$$\begin{aligned} a_{11} \cos 11\theta + b_{11} \sin 11\theta &= 3.83 \cos 11\theta - 3.31 \sin 11\theta \\ &= 5.07 \cos (11\theta + 41^\circ), \end{aligned}$$

hence, its effective value is

$$\frac{5.07}{\sqrt{2}} = 3.58,$$

while the effective value of the total generator wave, that is, the square root of the mean squares of the instantaneous values y, is

$$e = 30.5$$

thus the 11th harmonic is 11.8 per cent of the total voltage, and whether such a harmonic is safe or not, can now be determined from the circuit constants, more particularly its resistance.
82. In general, the successive harmonics decrease; that is, with increasing n, the values of a_n and b_n become smaller, and when calculating a_n and b_n by equation (18), for higher values of n they are derived as the small averages of a number of large quantities, and the calculation then becomes inconvenient and less correct.

Where the entire series of coefficients a_n and b_n is to be calculated, it thus is preferable not to use the complete periodic function y, but only the residual left after subtracting the harmonics which have already been calculated; that is, after a_0 has been calculated, it is subtracted from y, and the difference, $y_1 = y - a_0$, is used for the calculation of a_1 and b_1.

Then $a_1 \cos\theta + b_1 \sin\theta$ is subtracted from y_1, and the difference,

$$\begin{aligned} y_2 &= y_1 - (a_1 \cos\theta + b_1 \sin\theta) \\ &= y - (a_0 + a_1 \cos\theta + b_1 \sin\theta), \end{aligned}$$

is used for the calculation of a_2 and b_2.
Then $a_2 \cos 2\theta + b_2 \sin 2\theta$ is subtracted from y_2, and the rest, y_3, used for the calculation of a_3 and b_3, etc.

In this manner a higher accuracy is derived, and the calculation simplified by having the instantaneous values of the function of the same magnitude as the coefficients a_n and b_n.

As illustration, is given in Table III the calculation of the first three harmonics of the pulsating current, Fig. 41, Table I:
83. In electrical engineering, the most important periodic functions are the alternating currents and voltages. Due to the constructive features of alternating-current generators, alternating voltages and currents are almost always symmetrical waves; that is, the periodic function consists of alternate half-waves, which are the same in shape, but opposite in dircetion, or in other words, the instantaneous values from 180 deg . to 360 deg . are the same numerically, but opposite in sign, from the instantaneous values between 0 to 180 deg., and each cycle or period thus consists of two equal but opposite half cycles, as shown in Fig. 44. In the earlier days of electrical engineering, the frequency has for this reason frequently been expressed by the number of half-waves or alternations.

In a symmetrical wave, those harmonics which produce a difference in the shape of the positive and the negative halfwave, cannot exist; that is, their coefficients a and b must be zero. Only those harmonics can exist in which an increase of the angle θ by 180 deg ., or π, reverses the sign of the function. This is the case with $\cos n\theta$ and $\sin n\theta$, if n is an odd number. If, however, n is an even number, an increase of θ by π increases the angle $n\theta$ by 2π or a multiple thereof, thus leaves $\cos n\theta$ and $\sin n0$ with the same sign. The same applies to a_0. Therefore, symmetrical alternating waves comprise only the odd harmonics, but do not contain even harmonics or a constant term, and thus are represented by

$$\begin{aligned} y = a_1 \cos\theta \cdot + a_3 \cos 3\theta + a_5 \cos 5\theta + \ldots \\ + b_1 \sin\theta + b_3 \sin 3\theta + b_5 \sin 5\theta + \ldots \end{aligned} \tag{19}$$

When calculating the coefficients a_n and b_n of a symmetrical wave by the expression (18), it is sufficient to average from 0 to π; that is, over one half-wave only. In the second half-wave, $\cos n\theta$ and $\sin n\theta$ have the opposite sign as in the first half-wave, if n is an odd number, and since y also has the opposite sign in the second half-wave, $y \cos n\theta$ and $y \sin n\theta$ in the second half-wave traverses again the same values, with the same sign, as in the first half-wave, and their average thus is given by averaging over one half-wave only.

Therefore, a symmetrical univalent periodic function, as an

Table

θ	ν	$y_1 = y - \sigma_0$	$\mu_1 \cos\theta$	$y_1 \sin\theta$	$c_1 = a_1\cos\theta + b_1\sin\theta$	$\boldsymbol{y_2 = y_n - c_1}$
0	-60	-111	-111	0	-84	-27
10	-49	-100	-98	-17	-85	-15
20	-38	-89	-84	-30	-83	-6
30	-26	-77	-67	-38	-79	+2
40	-12	-63	-48	-40	-72	9
50	0	-51	-33	-39	-63	12
60	+11	-40	-20	-35	-52	12
70	27	-24	-8	-23	-40	16
80	39	-12	-2	-12	-26	14
90	50	-1	0	-1	-11	10
100	61	+10	-2	+10	+4	6
110	71	20	-7	+19	18	+2
120	81	30	-15	+26	32	-2
130	90	39	-25	+30	45	-6
140	99	48	-37	+31	58	-10
150	107	56	-49	+28	67	-11
160	114	63	-59	+22	75	-12
170	119	68	-67	+12	81	-13
180	122	71	-71	0	84	-13
190	124	73	-72	-13	85	-12
200	126	75	-71	-26	83	-8
210	125	74	-64	-37	79	-5
220	123	72	-55	-47	72	0
230	120	69	-44	-53	63	+6
240	116	65	-32	-28	52	13
250	110	59	-20	-56	40	19
260	100	49	-9	-48	26	23
270	85	34	0	-34	11	23
280	65	+14	+2	-14	-4	18
290	35	-16	-5	+15	-18	+2
300	+17	-34	-17	+30	-32	-2
310	0	-51	-33	+39	-45	-6
320	-13	-64	-49	+41	-58	-6
330	-26	-75.	-65	+37	-67	-8
340	-38	-89	-84	+30	-75	-14
350	-49	-100	-99	+17	-81	-19
Total . . + 1826 Divided by $36 \ldots + 50.7 = a_0$		Total. . . . . . - 1520 Divided by 18..... . $84.4 = a_1$		$-11.3 = b_1$	**Total**	

III.

$l_2 \cos 2\theta$	$y_2 \sin 2\theta$	$c_i = a_2 \cos 2\theta + b_2 \sin 2\theta$	$\nu_3 = y_2 - c_2$	$y_3 \cos 3\theta$	$\nu_3 \sin 3\theta$	θ
-27	0	-15	-12	-12	0	0
-14	-5	-12	-3	-3	-1	10
-5	-4	-7	+1	0	+1	20
+1	+2	-1	+3	0	+3	30
+2	+9	+4	+5	-2	+4	40
-2	+12	11	+1	-1	0	50
-6	+10	13	-1	+1	0	60
-12	+10	15	+1	-1	0	70
-13	+5	16	-2	+1	+2	80
-10	0	15	-5	0	+5	90
-6	-2	12	-6	-3	+5	100
-2	-1	7	-5	-4	+2	110
+1	+2	+1	-3	-3	0	120
+1	+6	-4	-2	-2	-1	130
-2	+10	-11	+1	0	+1	140
-5	+10	-13	+2	0	+2	150
-9	+8	-15	+3	-1	+3	160
-12	-4	-16	+3	-3	+1	170
-13	0	-15	+2	-2	0	180
-11	-4	-12	0	0	0	190
-6	-6	-7	-1	0	-1	200
-2	-4	-1	-4	0	-4	210
0	0	+4	-4	-2	-4	220
-1	+6	11	-5	-4	-2	230
-6	+11	13	0	0	0	240
-15	+12	15	+4	+4	+2	250
-22	+8	16	+7	+3	+6	260
-23	0	15	+8	0	+8	270
-17	-6	12	+6	-3	+5	280
-2	-1	7	-5	+4	-2	290
+1	+2	+1	-3	+3	0	300
+1	+6	-4	-2	+2	+1	310
-1	+6	-11	+5	-2	-4	320
-4	+7	-13	+5	0	-5	330
-11	+9	-15	+1	0	-1	340
-18	+6	-16	-3	-3	+1	350
-270	+120		Total.	-33	+27	
$-15.0 = a_2$	$+6.7 = b_2$	Divided by 18	. . . . . .	$-1.8 = a_3$	$+1.5 = b_3$	

alternating voltage and current usually is, can be represented by the expression,

$$\begin{aligned} y = a_1 \cos \theta + a_3 \cos 3\theta + a_5 \cos 5\theta + a_7 \cos 7\theta + \dots \\ + b_1 \sin \theta + b_3 \sin 3\theta + b_5 \sin 5\theta + b_7 \sin 7\theta + \dots \end{aligned} \tag{20}$$

where,

$$\left. \begin{aligned} a_1 &= 2 \text{ avg. } (y \cos \theta)_0{}^{\pi}; \quad & b_1 &= 2 \text{ avg. } (y \sin \theta)_0{}^{\pi}; \\ a_3 &= 2 \text{ avg. } (y \cos 3\theta)_0{}^{\pi}; \quad & b_3 &= 2 \text{ avg. } (y \sin 3\theta)_0{}^{\pi}; \\ a_5 &= 2 \text{ avg. } (y \cos 5\theta)_0{}^{\pi}; \quad & b_5 &= 2 \text{ avg. } (y \sin 5\theta)_0{}^{\pi}; \\ a_7 &= 2 \text{ avg. } (y \cos 7\theta)_0{}^{\pi}; \quad & b_7 &= 2 \text{ avg. } (y \sin 7\theta)_0{}^{\pi} \end{aligned} \right\}. \tag{21}$$

84. From 180 deg. to 360 deg., the even harmonics have the same, but the odd harmonics the opposite sign as from 0 to 180 deg. Therefore adding the numerical values in the range from 180 deg . to 360 deg . to those

> in the range from 0 to 180 deg ., the odd harmonics cancel, and only the even harmonics remain. Inversely, by subtracting, the even harmonics cancel, and the odd ones remain.

Hereby the odd and the even harmonics can be separated. If $y = y(\theta)$ are the numerical values of a periodic function from 0 to 180 deg ., and $y' = y(\theta + \pi)$ the numerical values of the same function from 180 deg. to 360 deg.,

$$y_2(\theta) = \frac{1}{2}\{y(\theta) + y(\theta + \pi)\} \tag{22}$$

is a periodic function containing only the even harmonics, and

$$y_1(\partial) = \frac{1}{2}\{y(\theta) - y(\theta + \pi)\} \tag{23}$$

is a periodic function containing only the odd harmonics; that is:

$$\begin{aligned} y_1(\theta) &= a_1 \cos\theta + a_3 \cos 3\theta + a_5 \cos 5\theta + \ldots \\ &\quad + b_1 \sin\theta + b_3 \sin 3\theta + b_5 \sin 5\theta + \ldots \qquad (24) \\ y_2(\theta) &= a_0 + a_2 \cos 2\theta + a_4 \cos 4\theta + \ldots \\ &\quad + b_2 \sin 2\theta + b_4 \sin 4\theta + \ldots \qquad (25) \end{aligned}$$

and the complete function is

$$y(\theta) = y_1(\theta) + y_2(\theta). \tag{26}$$

By this method it is convenient to determine whether even harmonics are present, and if they are present, to separate them from the odd harmonics.

Before separating the even harmonics and the odd harmonics, it is usually convenient to separate the constant term a_0 from the periodic function y, by averaging the instantaneous values of y from 0 to 360 deg . The average then gives a_0, and subtracted from the instantaneous values of y, gives

$$y_0(\theta) = y(\theta) - a_0 \tag{27}$$

as the instantaneous values of the alternating component of the periodic function; that is, the component y_0 contains only the trigonometric functions, but not the constant term. y_0 is ihen resolved into the odd series y_1, and the even series y_2.
85. The alternating wave y_0 consists of the cosine components:

$$u(\theta) = a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + a_4 \cos 4\theta + \ldots, \tag{28}$$

and the sine components:

$$v(\theta) = b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta + \ldots; \tag{29}$$

that is,

$$y_0(\theta) = u(\theta) + v(\theta). \tag{30}$$

The cosine functions retain the same sign for negative angles ($-\theta$), as for positive angles ($+\theta$), while the sine functions reverse their sign; that is,

$$u(-\theta) = +u(\theta) \quad \text{and} \quad v(-\theta) = -v(\theta). \tag{31}$$

Therefore, if the values of y_0 for positive and for negative angles θ are averaged, the sine functions cancel, and only the cosine functions remain, while by subtracting the values of y_0 for positive and for negative angles, only the sine functions remain; that is,

$$\left.\begin{aligned} y_0(\theta) + y_0(-\theta) &= 2u(\theta), \\ y_0(\theta) - y_0(-\theta) &= 2v(\theta); \end{aligned}\right\}. \tag{32}$$

hence, the cosine terms and the sine terms can be separated from each other by combining the instantaneous values of y_0 for positive angle θ and for negative angle ($-\theta$), thus:

$$\left.\begin{aligned} u(\theta) &= \tfrac{1}{2}\{y_0(\theta) + y_0(-\theta)\}, \\ v(\theta) &= \tfrac{1}{2}\{y_0(\theta) - y_0(-\theta)!\cdot \end{aligned}\right\}. \tag{33}$$

Usually, before separating the cosine and the sine terms, u and v, first the constant term a_0 is separated, as discussed above; that is, the alternating function $y_0 = y - a_0$ used. If the general periodic function y is used in equation (33), the constant term a_0 of this periodic function appears in the cosine term u, thus: $u(\theta) = \frac{1}{2}\{y(\theta) + y(-\theta)\} = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \ldots$, while $v(\theta)$ remains the same as when using y_0.

86. Before separating the alternating function y_0 into the cosine function u and the sine function v, it usually is more convenient to resolve the alternating function y_0 into the odd series y_1, and the even series y_2, as discussed in the preceding paragraph, and then to separate y_1 and y_2 each into the cosine and the sine terms:

$$\left.\begin{aligned} u_1(\theta) &= \tfrac{1}{2}\{y_1(\theta) + y_1(-\theta)\} = a_1 \cos\theta + a_3 \cos 3\theta + a_5 \cos 5\theta + \ldots; \\ v_1(\theta) &= \tfrac{1}{2}\{y_1(\theta) - y_1(-\theta)\} = b_1 \sin\theta + b_3 \sin 3\theta + b_5 \sin 5\theta + \ldots; \end{aligned}\right\} \quad (34)$$

$$\left.\begin{aligned} u_2(\theta) &= \tfrac{1}{2}\{y_2(\theta) + y_2(-\theta)\} = a_2 \cos 2\theta + a_4 \cos 4\theta + \ldots \\ v_2(\theta) &= \tfrac{1}{2}\{y_2(\theta) - y_2(-\theta)\} = b_2 \sin 2\theta + b_4 \sin 4\theta + \ldots \end{aligned}\right\}. \quad (35)$$

In the odd functions u_1 and v_1, a change from the negative angle (-8) to the supplementary angle ($\pi - \theta$) changes the angle of the trigonometric function by an odd multiple of π or 180 deg., that is, by a multiple of 2π or 360 deg., plus 180 deg., which signifies a reversal of the function, thus:

$$\left.\begin{aligned} u_1(\theta) &= \tfrac{1}{2}\{y_1(\theta) - y_1(\pi - \theta)\}, \\ v_1(\theta) &= \tfrac{1}{2}\{y_1(\theta) + y_1(\pi - \theta)\}. \end{aligned}\right\}. \quad (36)$$

However, in the even functions u_2 and v_2 a change from the negative angle ($-\theta$) to the supplementary angle ($\pi - \theta$), changes the angles of the trigonometric function by an even multiple of π; that is, by a multiple of 2π or 360 deg .; hence leaves the sign of the trigonometric function unchanged, thus:

$$\left.\begin{aligned} u_2(\theta) &= \tfrac{1}{2}\{y_2(\theta) + y_2(\pi - \theta)\}, \\ v_2(\theta) &= \tfrac{1}{2}\{y_2(\theta) - y_2(\pi - \theta)\}. \end{aligned}\right\}. \quad (37)$$

To avoid the possibility of a mistake, it is preferable to use the relations (34) and (35), which are the same for the odd and for the even series.

87. Obviously, in the calculation of the constants a_n and b_n, instead of averaging from 0 to 180 deg., the average can be made from -90 deg. to +90 deg. In the cosine function $u(\theta)$, however, the same numerical values repeated with the same signs, from 0 to -90 deg., as from 0 to +90 deg., and the multipliers $\cos n\theta$ also have the same signs and the same numerical values from 0 to -90 deg., as from 0 to +90 deg. In the sine function, the same numerical values repeat from 0 to -90 deg ., as from 0 to +90 deg ., but with reversed signs, and the multipliers $\sin n\theta$ also have the same numerical values, but with reversed sign, from 0 to -90 deg., as from 0 to +90 deg. The products $u \cos n\theta$ and $v \sin n\theta$ thus traverse the same numerical values with the same signs, between 0 and -90 deg ., as between 0 and +90 deg ., and for deriving the averages, it thus is sufficient to average only from 0 to $\frac{\pi}{2}$, or 90 deg .; that is, over one quandrant.

Therefore, by resolving the periodic function y into the cosine components u and the sine components v, the calculation of the constants a_n and b_n is greatly simplified; that is, instead of averaging over one entire period, or 360 deg., it is necessary to average over only 90 deg ., thus:

$$\left.\begin{aligned} a_1 &= 2 \text{ avg. } (u_1 \cos\theta)_0^{\frac{\pi}{2}}; & b_1 &= 2 \text{ avg. } (v_1 \sin\theta)_0^{\frac{\pi}{2}}; \\ a_2 &= 2 \text{ avg. } (u_2 \cos 2\theta)^{\frac{\pi}{2}}; & b_2 &= 2 \text{ avg. } (v_2 \sin 2\theta)_0^{\frac{\pi}{2}}; \\ a_3 &= 2 \text{ avg. } (u_3 \cos 3\theta)_0^{\frac{\pi}{2}}; & b_3 &= 2 \text{ avg. } (v_3 \sin 3\theta)_0^{\frac{\pi}{2}}; \\ a_4 &= 2 \text{ avg. } (u_4 \cos 4\theta)_0^{\frac{\pi}{2}}; & b_4 &= 2 \text{ avg. } (v_4 \sin 4\theta)_0^{\frac{\pi}{2}}; \\ a_5 &= 2 \text{ avg. } (u_5 \cos 5\theta)_0^{\frac{\pi}{2}}; & b_5 &= 2 \text{ avg. } (v_5 \sin 5\theta)_0^{\frac{\pi}{2}}; \\ &\text{etc.} & &\text{etc.} \end{aligned}\right\}. \quad (38)$$

where u_1 is the cosine term of the odd function y_1; u_2 the cosine term of the even function y_2; u_3 is the cosine term of the odd function, after subtracting the term with $\cos\theta$; that is,

$$u_3 = u_1 - a_1 \cos\theta,$$

analogously, u_4 is the cosine term of the even function, after subtracting the term $\cos 2\theta$;

$$u_4 = u_2 - a_2 \cos 2\theta$$

and in the same manner,

$$u_5 = u_3 - a_3 \cos 3\theta$$
$$u_6 = u_4 - a_4 \cos 4\theta$$

and so forth; v_1, v_2, v_5, v_4, etc., are the corresponding sine terms.

When calculating the coefficients a_n and b_n by averaging over 90 deg ., or over 180 deg . or 360 deg ., it must be kept in mind that the terminal values of y respectively of u or v, that is, the values for $\theta = 0$ and $\theta = 90$deg. (or $\theta = 180$deg. or 360 deg. respectively) are to be taken as one-half only, since they are the ends of the measured area of the curves $a_n \cos n\theta$ and $b_n \sin n\theta$, which area gives as twice its average height the values a_n and b_n, as discussed in the preceding.

In resolving an empirical periodic function into a trigonometric series, just as in most engineering calculations, the most important part is to arrange the work so as to derive the results expeditiously and rapidly, and at the same time accurately. By proceeding, for instance, immediately by the general method, equations (17) and (18), the work becomes so extensive as to be a serious waste of time, while by the systematic resolution into simpler functions the work can be greatly reduced.

88. In resolving a general periodic function $y(\theta)$ into a trigonometric series, the most convenient arrangement is:

1. To separate the constant term a_0, by averaging all the instantaneous values of $y(\theta)$ from 0 to 360 deg . (counting the end values at $\theta = 0$ and at $\theta = 360$deg. one half, as discussed above):

$$a_0 = \text{avg.}\,\{y(\theta)\}_0^{2\pi} \tag{10}$$

and then subtracting a_0 from $y(0)$, gives the alternating function,

$$y_0(\theta) = y(\theta) - a_0.$$

2. To resolve the general alternating function $y_0(\theta)$ into the odd function $y_1(\theta)$, and the even function $y_2(\theta)$,

$$y_1(\theta) = \frac{1}{2}\{y_0(\theta) - y_0(\theta + \pi)\}; \tag{23}$$
$$y_2(\theta) = \frac{1}{2}\{y_0(\theta) + y_0(\theta + \pi)\}. \tag{22}$$

3. To resolve $y_1(\theta)$ gnd $y_2(\theta)$) into the cosine terms u and the sine terms v,

$$\left.\begin{aligned} u_1(\theta) &= \tfrac{1}{2}\{y_1(\theta) + y_1(-\theta)\}; \\ v_1(\theta) &= \tfrac{1}{2}\{y_1(\theta) - y_1(-\theta)\}; \end{aligned}\right\}, \ldots\ldots \tag{35}$$

4. To calculate the constants $a_1, a_2, a_3 \ldots ; b_1, b_2, b_3 \ldots$ by the averages,

$$\left.\begin{aligned} a_n &= 2\,\text{avg}\cdot(u_n \cos n\theta)_0^{\frac{\pi}{2}} \\ b_n &= 2\,\text{avg}\cdot(v_n \sin n\theta)_0^{\frac{\pi}{2}} \end{aligned}\right\}. \tag{38}$$

If the periodic function is known to contain no even harmonies, that is, is a symmetrical alternating wave, steps 1 and 2 are omitted.

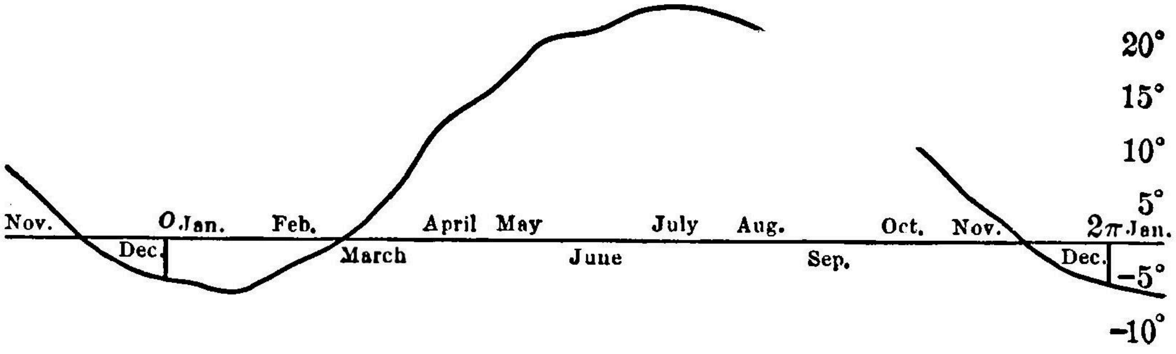

Fig. 45. Mean Daily Temperature at Schenectady.

89. As illustration of the resolution of a general periodic wave may be shown the resolution of the observed mean daily temperatures of Schenectady throughout the year, as shown in Fig. 45, up to the 7th harmonic.*

Table IV

(1) θ	(2) y	(3) $y\ y - a_0 = y_0$	(4) y_1	(5) y_2
Jan. 0	- 4.2	-12.95	-13.10	+0.15
10	- 4.7	-13.45	-13.55	+0.10
20	- 5.2	-13.95	-13.65	-0.30
Feb. 30	- 5.4	-14.15	-13.55	-0.60
40	- 3.8	-12.55	-12.35	-0.20
50	- 2.6	-11.35	-11.20	-0.15
Mar. 60	- 1.6	-10.35	- 9.75	-0.60
70	+ 0.2	- 8.55	- 7.65	-0.90
80	+ 1.8	- 6.95	- 6.05	-0.90
Apr. 90	+ 5.1	- 3.65	- 3.35	-0.30
100	+ 9.1	+ 0.35	- 0.35	+0.70
110	+11.5	+ 2.75	+ 1.75	+1.00
May 120	+13.3	+ 4.55	+ 3.90	+0.65
130	+15.2	+ 6.45	+ 5.85	+0.60
140	+17.7	+ 8.95	+ 8.15	+0.80
June 150	+19.2	+10.45	+10.10	+0.35
160	+19.5	+10.75	+10.80	-0.05
170	+20.6	+11.85	+12.15	-0.30
July 180	+22.0	+13.25		
190	+22.4	+13.65		
200	+22.1	+13.35		
Aug. 210	+21.7	+12.95		
220	+20.9	+12.15		
230	+19.8	+11.05		
Sept. 240	+17.9	+ 9.15		
250	+15.5	+ 6.75		
260	+13.8	+ 5.15		
Oct. 270	+11.8	+ 3.05		
280	+ 9.8	+ 1.05		
290	+ 8.0	- 0.75		
Nov. 300	+ 5.5	- 3.25		
310	+ 3.5	- 5.25		
320	+ 1.4	- 7.35		
Dec. 330	- 1.0	- 9.75		
340	- 2.1	-10.85		
350	- 3.7	-12.45		
Total	315.1			
Divided by 36	$8.75 = a_0$			

Table V.

(1) 0	(2) V1	(3) u1	(4) v_1	(5) y 2	(6)	(7) $\boldsymbol{v_2}$
-90	+ 3.35			-0.30		
-80	+ 0.35			+0.70		
-70	- 1.75			+1.00.		
-60	- 3.90			+0.65		
-50	- 5.85			+0.60		
-40	- 8.15			+0.80		
-30	-10.10			+0.35		
-20	-10.80			-0.05		
-10	-12.15			-0.30		
0	-13.10	-13.10	0	+0.15	+0.15	0
+10	-13.55	-12.85	-0.70	+0.10	-0.10	+0.20
+20	-13.65	-12.23	-1.42	-0.30	-0.17	-0.12
+30	-13.55	-11.82	-1.73	-0.60	-0.12	-0.47
+40	-12.35	-10.25	-2.10	-0.20	+0.30	-0.50
+50	-11.20	- 8.53	-2.67	-0.15	+0.22	-0.37
+60	- 9.75	- 6.82	-2.93	-0.60	+0.02	-0.62
+70	- 7.65	- 4.70	-2.95	-0.90	+0.05	-0.95
+80	- 6.05	- 2.85	-3.20	-0.90	-0.10	-0.80
+90	- 3.35	0	-3.35	-0.30	-0.30	0

Table VI. COSINE SERIES u_1.

(1) θ	(2) u_1	(3) $\cos\theta$	(4) $u1 \cos\theta$	(5) $a_1 \cos\theta$	(6) u_3	(7) u_3 **cos 30**	(8) $a_3 \cos 30$	(9) u_5	(10) $u5 \cos 50$	(11) $u5$ c
0	-13.10	1	$-13.10\left(\times\frac{1}{2}\right)$	-13.28	+0.18	$+0.18\left(\times\frac{1}{2}\right)$	+0.33	-0.15	DE ($\times\frac{1}{2}$)	-0.15 (
10	-12.85	0.985	-12.65	-13.05	-0.20	+0.17	0.285	-0.085	An54	-0.029
20	-12.23	0.940	-11.50	-12.48	+0.25	+0.12	0.165	+0.085	0-015	-0.065
30	-11.82	0.866	-10.25	-11.50	-0.32	0	0	-032	+0.277	-0.239
40	-10.25	0.766	-7.83	-10.15	-0.10	+0.05	-0.165	+0.065	0-061	-0.011
50	-8.53	0.643	-5.46	-8.00	-0.53	+0.46	-0.285	-0.245	θα4	+0.083
60	-6.82	0.5	-3.41	-6.64	-0.18	+0.18	-0.33	+0.15	+0.075	+0.037
70	- 9.70	0.342	-1.61	-4.54	-0.16	+0.14	-0.285	+0.125	+0.123	-0.079
80	2.85	0.174	-0.50	-2.30	-0.55	+0.27	-0.165	-0.385	-0.293	+0.276
90	0	0	0	0	0	0	0	0	0	0
Total			-59.75			+1.48			+0.061	-0.101
Divided by 9			-6.64			+1.64			+0.0068	-0.011
Multiplied by 2			$-13.28 = a_1$			$+0.33 = a_5$			$+0.014 = a_5$	-0.022

Table VII. SINE SERIES v_1.

(1) θ	(2) v_1	(3) $\sin\theta$	(4) $v1 \sin\theta$	(5) $b_1 \sin\theta$	(6) $\boldsymbol{v_3}$	(7) $v_3 \sin 3\theta$	(8) $b_3 \sin 3\theta$	(9) vs	(10) $v5 \sin 5\theta$	(11) $v5 \sin 7\theta$
0	0	0	0	0	0	0	0	0	0	0
10	-0.70	0.174	-0.122	-0.38	-0.32	-0.16	-0.07	-0.25	-0.19	-0.23
20	-1.42	0.342	-0.485	-1.14	-0.28	-0.24	-0.12	-0.16	-0.16	-0.10
30	-1.73	0.5	-0.865	-1.66	-0.07	-0.67	-0.14	+0.07	+0.035	-0.035
40	-2.10	0.643	-1.345	-2.13	+0.03	+0.03	-0.12	0.15	-0.05	-0.15
50	-2.67	0.766	-2.040	-2.55	-0.12	-0.06	-0.07	-0.05	+0.05	+0.01
60	-2.93	0.866	-2.530	-2.88	-0.05	0	0	-0.05	+0.04	-0.04
70	-2.95	0.940	-2.770	-3.13	+0.18	-0.09	+0.07	+0.11	-0.02	+0.085
80	-3.20	0.985	-3.150	-3.28	+0.08	-0.07	+0.12	-0.04	-0.025	+0.01
90	-3.35	1	$-\mathbf{3.35}\left(\times\frac{1}{2}\right)$	-3.33	-0.02	$+0.02\left(\times\frac{1}{2}\right)$	+0.14	-0.16	$-0.16\left(\times\frac{1}{2}\right)$	$+0.16\left(\times\frac{1}{2}\right)$
Total			-14.982			-0.65			-0.40	-0.37
Divided by 9			-1.665			-0.07			-0.044	-0.041
Multiplied by 2			$-\mathbf{3.33} = b_1$			$-0.14 = b_3$			$-0.09 = b_5$	$-0.082 = b_7$

Table VIII.
COSINE SERIES u_2.

(1) θ	(2) u_2	(3) $u_2 \cos 2\theta$	(4) $a_2 \cos 2\theta$	(5) u_4	(6) $u_4 \cos 4\theta$	(7) $a_4 \cos 4\theta$	(8) u_6	$u_6 \cos 6\theta$
0	+0.15	$\frac{1}{2}(+0.15)$	0	+0.15	$\frac{1}{2}(+0.15)$	-0.16	+0.31	$\frac{1}{2}(+0.31)$
10	-0.10	-0.09	.....	-0.10	-0.08	-0.12	+0.02	+0.01
20	-0.17	-0.13	.....	-0.17	-0.03	-0.03	-0.14	+0.07
30	-0.12	-0.06		-0.12	+0.06	+0.08	-0.20	+0.20
40	+0.30	+0.05		+0.30	-0.29	+0.15	+0.15	-0.07
50	+0.22	-0.04		+0.22	-0.21	+0.15	+0.07	+0.03
60	+0.02	-0.01		+0.02	-0.01	+0.08	-0.06	-0.06
70	+0.05	-0.04		+0.05	+0.01	-0.03	+0.08	+0.04
80	-0.10	+0.09		-0.10	-0.08	-0.12	+0.02	-0.01
90	-0.30	$\frac{1}{2}(+0.30)$	0	-0.30	$\frac{1}{2}(+0.30)$	-0.16	-0.14	$\frac{1}{2}(+0.14)$
Total		-0.01			-0.71			+0.44
9 9...... .		-0.001			-0.079			+0.049
by 2		-0.002			-0.158			+0.098
		$= a_2$			$= a_3$			$= a_0$

Table IX.
SINE SERIES v_2.

(1) θ	(2) v_2	(3) $v_2 \sin 2\theta$	(4) $b_2 \sin 2\theta$	(5) v4	(6) $v4 \sin 4\theta$	(7) $b_4 \sin 4\theta$	(8)	(9) $v_6 \sin 6\theta$
0	0	0						
10	+0.20	+0.07	-0.20	+0.40	+0.26	+0.22	+0.18	+0.16
20	-0.12	-0.08	-0.39	+0.27	+0.27	+0.34	-0.07	-0.07
30	-0.47	-0.41	-0.52	+0.05	+0.04	+0.30	-0.25	+0
40	-0.50	-0.49	-0.59	+0.09	+0.03	+0.12	-0.03	+0.03
50	-0.37	-0.36	-0.59	+0.22	-0.08	-0.12	+0.34	-0.30
60	-0.62	-0.54	-0.52	-0.10	+0.09	-0.30	+0.20	0
70	-0.95	-0.61	-0.39	-0.56	+0.55	-0.34	-0.22	-0.19
80	-0.80	-0.27	-0.20	-0.60	+0.39	+0.22	-0.38	-0.33
	Total…….	-2.69			+1.55			-0.70
	Divided by 9	-0.30			+0.172			-0.078
	Divided by 2	-0.60			+0.344			-0.156
		$= b_2$			$= b_4$			$= b_6$

Table IV gives the resolution of the periodic temperature function into the constant term a_0, the odd series y_1 and the even series y_2.

Table V gives the resolution of the series y_1 and y_2 into the cosine and sine series u_1, v_1, u_2, v_2.

Tables VI to IX give the resolutions of the series u_1, v_1, u_2, v_2, and thereby the calculation of the constants a_n and b_n.

90. The resolution of the temperature wave, up to the 7th harmonic, thus gives the coefficients:

$$\begin{aligned} a_0 &= +8.75; \\ a_1 &= -13.28; \quad b_1 = -3.33; \\ a_2 &= -0.001; \quad b_2 = -0.602; \\ a_3 &= -0.33; \quad b_3 = -0.14; \\ a_4 &= -0.154; \quad b_4 = +0.386; \\ a_5 &= +0.014; \quad b_5 = -0.090; \\ a_6 &= +0.100; \quad b_6 = -0.154; \\ a_7 &= -0.022; \quad b_7 = -0.082; \end{aligned}$$

or, transforming by the binomial, $a_n \cos n\theta + b_n \sin n\theta = c_n \cos(n\theta - \gamma_n)$, by substituting $c_n = \sqrt{a_n^2 + b_n^2}$ and $\tan \gamma_n = \frac{b_n}{a_n}$ gives, $a_0 = +8.75$;
$c_1 = -13.69; \quad \gamma_1 = +14.15^\circ$; or $\gamma_1 = +14.15^\circ$;
$c_2 = -0.602; \quad r_2 = +89.9^\circ; \quad$ or $\frac{r_2}{2} = +44.95^\circ + 180n$;
$c_3 = +0.359; \quad \gamma_3 = -23.0^\circ; \quad$ or $\frac{\gamma_3}{3} = -7.7 + 120n = +112.3 + 120m$;
$c_4 = -0.416; \quad \gamma_4 = -68.2^\circ; \quad$ or $\frac{\gamma_4}{4} = -17.05 + 90n = +72.95 + 90m$;
$c_5 = +0.091; \quad \gamma_5 = -81.15^\circ$; or $\frac{\gamma_5}{5} = -16.23 + 72n = +55.77 + 72m$;
$c_6 = +0.184; \quad \gamma_6 = -57.0^\circ; \quad$ or $\frac{\gamma_6}{6} = -9.5 + 60n = +50.5 + 60m$;
$c_7 = -0.085; \quad \gamma_7 = +75.0^\circ; \quad$ or $\frac{\gamma_7}{7} = +10.7 + 51.4n$,
where n and m may be any integer number.

Since to an angle γ_n, any multiple of 2π or 360 deg. may be added, any multiple of $\frac{360}{n}$ may be added to the angle $\frac{r_n}{n}$, and thus the angle $\frac{\gamma_n}{n}$ may be made positive, etc.

91. The equation of the temperature wave thus becomes:

$$\begin{aligned} y = 8.75 &- 13.69 \cos(\theta - 14.15^\circ) - 0.602 \cos 2(\theta - 44.95^\circ) \\ &- 0.359 \cos 3(\theta - 52.3^\circ) - 0.416 \cos 4(\theta - 72.95^\circ) \\ &- 0.091 \cos 5(\theta - 19.77^\circ) - 0.184 \cos 6(\theta - 20.5^\circ) \\ &- 0.085 \cos 7(\theta - 10.7^\circ); \end{aligned} \tag{a}$$

or, transformed to sine functions by the substitution,

$$\begin{gathered}\cos \omega = -\sin (\omega - 90^\circ) : \\ y = 8.75 + 13.69 \sin (\theta - 104.15^\circ) + 0.602 \sin 2 (\theta - 89.95^\circ) \\ +0.359 \sin 3 (\theta - 82.3^\circ) + 0.416 \sin 4 (\theta - 95.45^\circ) \\ +0.091 \sin 5 (\theta - 109.77^\circ) + 0.184 \sin 6 (\theta - 95.5^\circ) \\ +0.085 \sin 7 (\theta - 75^\circ). \end{gathered} \tag{b}$$

The cosine form is more convenient for some purposes, the sine form for other purposes.

Substituting $\beta = \theta - 14.15^\circ$; or, $\delta = \theta - 104.15^\circ$, these two equations (a) and (b) can be transformed into the form,

$$\begin{aligned} y = 8.75 &- 13.69 \cos \beta - 0.62 \cos 2 (\beta - 30.8^\circ) - 0.359 \cos 3 (\beta - 38.15^\circ) \\ &- 0.416 \cos 4 (\beta - 58.8^\circ) - 0.091 \cos 5 (\beta - 5.6^\circ) \\ &- 0.184 \cos 6 (\beta - 6.35^\circ) - 0.085 \cos 7 (\beta - 48.0^\circ), \end{aligned} \tag{c}$$

and

$$\begin{aligned} y = 8.75 &+ 13.69 \sin \delta + 0.602 \sin 2 (\delta + 14.2^\circ) + 0.359 \sin 3 (\delta + 21.85^\circ) \\ &+ 0.416 \sin 4 (\delta + 8.7^\circ) + 0.91 \sin 5 (\delta - 5.6^\circ) \\ &+ 0.184 \sin 6 (\delta + 8.65^\circ) + 0.085 \sin 7 (\delta + 29.15^\circ). \end{aligned} \tag{d}$$

The periodic variation of the temperature y, as expressed by these equations, is a result of the periodic variation of the thermomotive force; that is, the solar radiation. This latter is a minimum on Dec. 22d, that is, 9 time-degrees before the zero of θ, hence may be expressed approximately by:

$$z = c - h \cos (\theta + 9^\circ)$$

or substituting β respectively δ for θ :

$$\begin{gathered} z = c - h \cos (\beta + 23.15^\circ) \\ . = c + h \sin (\delta + 23.15^\circ). \end{gathered}$$

This means: the maximum of y occurs 23.15 deg . after the naximum of z; in other words, the temperature lags 23.15 deg ., or about $\frac{1}{16}$ period, behind the thermomotive force.

Near $\delta = 0$, all the sine functions in (d) are increasing; that is, the temperature wave rises steeply in spring.

Near $\delta = 180$deg., the sine functions of the odd angles are decreasing, of the even angles increasing, and the decrease of the temperature wave in fall thus is smaller than the increase in spring.

The fundamental wave greatly preponderates, with amplitude $c_1 = 13.69$.

In spring, for $\delta = -14.5$ deg., all the higher harmonics rise in the same direction, and give the sum 1.74 , or 12.7 per cent of the fundamental. In fall, for $\delta = -14.5 + \pi$, the even harmonics decrease, the odd harmonics increase the steepness, and give the sum -0.67 , or -4.9 per cent.

Therefore, in spring, the temperature rises 12.7 per cent faster, and in autumn it falls 4.9 per cent slower than corresponds to a sine wave, and the difference in the rate of temperature rise in spring, and temperature fall in autumn thus is $12.7 + 4.9 = 17.6$ per cent.

The maximum rate of temperature rise is $90 - 14.5 = 75.5$ deg. behind the temperature minimum, and $23.15 + 75.5 = 98.7$ deg . behind the minimum of the thermomotive force.

As most periodic functions met by the electrical engineer are symmetrical alternating functions, that is, contain only the odd harmonics, in general the work of resolution into a trigonometric series is very much less than in above example. Where such reduction has to be carried out frequently, it is advisable to memorize the trigonometric functions, from 10 to 10 deg., up to 3 decimals; that is, within the accuracy of the slide rule, as thereby the necessity of looking up tables is eliminated and the work therefore done much more expeditiously. In general, the slide rule can be used for the calculations.

As an example of the simpler reduction of a symmetrical alternating wave, the reader may resolve into its harmonics, up to the 7th, the exciting current of the transformer, of which the numerical values are given, from 10 to 10 deg . in Table X.

C. REDUCTION OF TRIGONOMETRIC SERIES BY POLYPHASE RELATION.

92. In some cases the reduction of a general periodic function, as a complex wave, into harmonics can be carried out in a much quicker manner by the use of the polyphase equation, Chapter III, Part A (23). Especially is this true if the complete equation of the trigonometric series, which represents the periodic function, is not required, but the existence and the amount of certain harmonics are to be determined, as for instance whether the periodic function contain even harmonics or third harmonics, and how large they may be.

This method does not give the coefficients a_n, b_n of the individual harmonics, but derives from the numerical values of the general wave the numerical values of any desired harmonic. This harmonic, however, is given together with all its multiples; that is, when separating the third harmonic, in it appears also the 6th, 9th, 12th, etc.

In separating the even harmonics y_2 from the general wave y, in paragraph 84 , by taking the average of the values of y for angle θ, and the values of y for angles $(\theta+\pi)$, this method has already been used.

Assume that to an angle 0 there is successively added a constant quantity a, thus: $\theta ; \theta+a ; \theta+2 a ; \theta+3 a ; \theta+4 a$, etc., until the same angle θ plus a multiple of 2π is reached; $\theta+n a=\theta+2 m \pi$; that is, $a=\frac{2 m \pi}{n}$; or, in other words, a is $1 / n$ of a multiple of 2π. Then the sum of the cosine as well as the sine functions of all these angles is zero:

$$\begin{aligned}\cos \theta+\cos (\theta+a)+\cos (\theta+2 a)+\cos (\theta+3 a)+\ldots \\ +\cos (\theta+[n-1] a)=0 ;\end{aligned} \tag{1}$$

$$\begin{aligned}\sin \theta+\sin (\theta+a)+\sin (\theta+2 a)+\sin (\theta+3 a)+\ldots \\ +\sin (\theta+[n-1] a)=0,\end{aligned} \tag{2}$$

where

$$n a=2 m \pi \tag{3}$$

These equations (1) and (2) hold for all values of a, except for $a=2 \pi$, or a multiple thereof. For $a=2 \pi$ obviously all the terms of equation (1) or (2) become equal, and the sums become $n \cos \theta$ respectively $n \sin \theta$.

Thus, if the series of numerical values of y is divided into n successive sections, each covering $\frac{2 \pi}{n}$ degrees, and these sections added together,

$$\begin{aligned}y(\theta)+y\left(\theta+\frac{2 \pi}{n}\right)+y\left(\theta+2 \frac{2 \pi}{n}\right)+y\left(\theta+3 \frac{2 \pi}{n}\right)+\ldots \\ +y\left(\theta+[n-1] \frac{2 \pi}{n}\right), \\ ..\end{aligned} \tag{4}$$

In this sum, all the harmonics of the wave y cancel by equations (1) and (2), except the nth harmonic and its multiples,

$$a_n \cos n \theta+b_n \sin n \theta ; a_{2 n} \cos 2 n \theta+b_{2 n} \sin 2 n \theta, \text { etc. }$$

in the latter all the terms of the sum (4) are equal; that is, the sum (4) equals n times the nth harmonic, and its multiples. Therefore, the nth harmonic of the periodic function y, together with its multiples, is given by

$$y_n(\theta)=\frac{1}{n}\left\{y(\theta)+y\left(\theta+\frac{2 \pi}{n}\right)+y\left(0+2 \frac{2 \pi}{n}\right)+\ldots+y\left(0+[n-1] \frac{2 \pi}{n}\right)\right\} \tag{5}$$

For instance, for $n=2$,

$$y_2=\frac{1}{2}\{y(\theta)+y(\theta+\pi)\}$$

gives the sum of all the even harmonics; that is, gives the second harmonic together with its multiples, the 4th, 6th, etc., as seen in paragraph 7, and for, $n=3$,

$$y_3=\frac{1}{3}\left\{y(0)+y\left(0+\frac{2 \pi}{3}\right)+y\left(0+\frac{4 \pi}{3}\right)\right\},$$

gives the third harmonic, together with its multiples, the 6th, 9th, etc.

This method does not give the mathematical expression of the harmonics, but their numerical values. Thus, if the mathematical expressions are required, each of the component harmonics has to be reduced from its numerical values to the mathematical equation, and the method then usually offers no advantage.

It is especially suitable, however, where certain classes of harmonics are desired, as the third together with its multiples. In this case from the numerical values the effective value, that is, the equivalent sine wave may be calculated.
93. As illustration may be investigated the separation of the third harmonics from the exciting current of a transformer.

Table X

A						
(1) θ	(2) i	(3) θ	(4)	(5) θ	(6)	(7)
0	+24.0	120	-15.1	240	+8.5	+5.8
10	+20.0	130	-16.5	250	+10	+4.5
20	+ 12	140	-18.5	260	+11	+1.5
30	+ 4	150	-21	270	+12	-1.7
40	- 1.5	160	-22.7	280	+13	-3.7
50	- 6.5	170	-23.7	290	+14	-5.4
60	- 8.5	180	-24	300	+15.1	-5.8
B						
30	i_3	30	i_3	3θ	i_3	i_9
0	+5.8	120	-3.7	240	-1.5	+0.2
30	+4.5	150	-5.4	270	+1.7	+0.3
60	+1.5	180	-5.8	300	+3.7	-0.2

In table X A , are given, in columns $1,3,5$, the angles θ, from 10 deg . to 10 deg ., and in columns $2,4,6$, the corresponding values of the exciting current i, as derived by calculation from the hysteresis cycle of the iron, or by measuring from the photographic film of the oscillograph. Column 7 then gives one-third the sum of columns 2,4 , and 6 , that is, the third harmonic with its overtones, i_3.

To find the 9th harmonic and its overtones i_9, the same method is now applied to i_3, for angle 3θ. This is recorded in Table X B.

In Fig. 46 are plotted the total exciting current i, its third harmonic i_3, and the 9 th harmonic i_9.

This method has the advantage of showing the limitation of the exactness of the results resulting from the limited num-

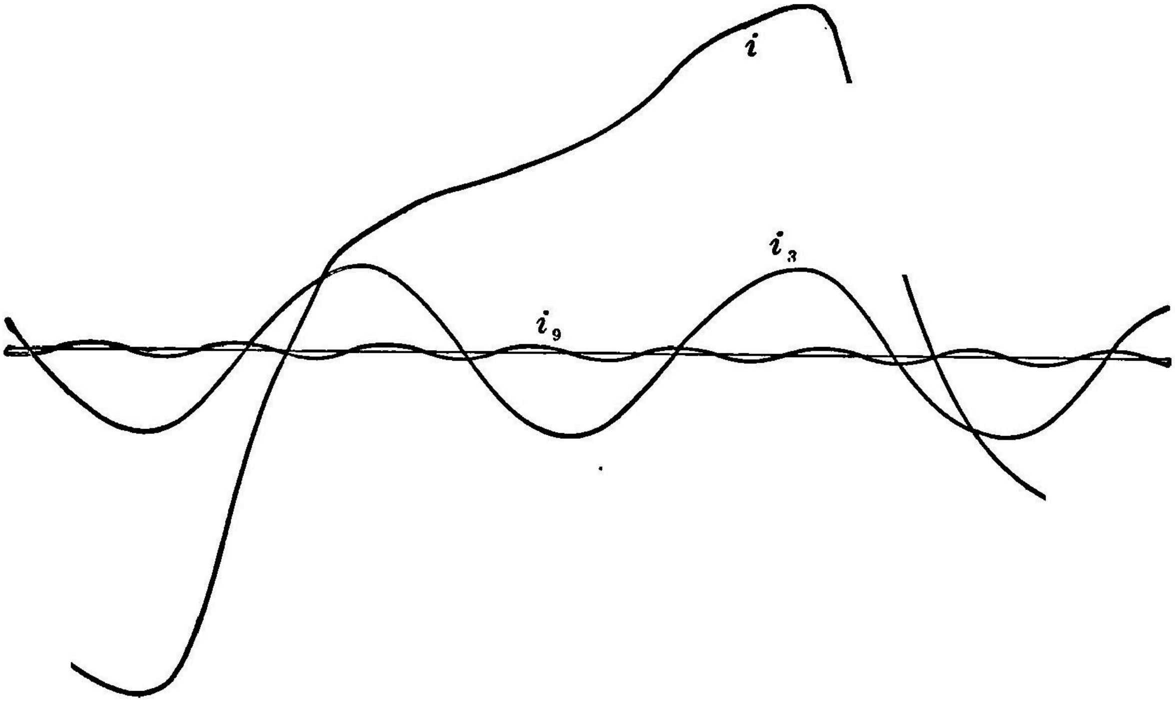

Fig. 46.

ber of numerical values of i, on which the calculation is based. Thus, in the example, Table X , in which the values of i are given for every 10 deg ., values of the third harmonic are derived for every 30 deg ., and for the 9 th harmonic for every 90 deg .; that is, for the latter, only two points per half wave are determinable from the numerical data, and as the two points per half wave are just sufficient to locate a sine wave, it follows that within the accuracy of the given numerical values of i, the 9th harmonic is a sine wave, or in other words, to determine whether still higher harmonics than the 9th exist, requires for i more numerical values than for every 10 deg .

As further practice, the reader may separate from the general wave of current, i_0 in Table XI, the even harmonics i_2, by above method,

$$i_2=\frac{1}{2}\{i_0(\theta)+i_0(\theta+180\text{deg.})\},$$

and also the sum of the odd harmonics, as the residue,

$$i_1=i_0-i_2,$$

then from the odd harmonics i_1 may be separated the third harmonic and its multiples,

$$i_3=\frac{1}{3}\{i_1(\theta)+i_1(\theta+120\text{ deg. })+i_1(\theta+240\text{ deg. })\},$$

and in the same manner from i_3 may be separated its third harmonic; that is, i_9.

Furthermore, in the sum of even harmonics, from i_2 may again be separated its second harmonic, i_4, and its multiples, and therefrom, i_8, and its third harmonic, i_6, and its multiples, thus giving all the harmonics up to the 9th, with the exception of the 5th and the 7th. These latter two would require plotting the curve and taking numerical values at different intervals, so as to have a number of numerical values divisible by 5 or 7 .

It is further recommended to resolve this unsymmetrical exciting current of Table XI into the trigonometric series by calculating the coefficients a_n and b_n, up to the 7th, in the manner discussed in paragraphs 6 to 8 .

Table XI

θ	i_0	θ	i_0	θ	i_0	θ	io
0	+95.7	90	-26.7	180	-34.3	270	- 3.3
10	+78.7	100	-27.3	190	-27.3	280	- 1.8
20	+53.7	110	-28.1	200	-16.8	290	+ 1.2
30	+23.7	120	-28.8	210	-11.3	300	+ 4.7
40	- 2.3	130	-29.3	220	- 8.3	310	+10.7
50	-16.3	140	-29.8	230	- 7.3	320	+22.7
60	-22.8	150	-31	240	- 6.3	330	+41.7
70	-24.3	160	-32.6	250	- 5.3	340	+65.7
80	-25.8	170	-33.8.	260	- 43	350	+85.7

D. CALCULATION OF TRIGONOMETRIC SERIES FROM OTHER TRIGONOMETRIC SERIES.

94. An hydraulic generating station has for a long time been supplying electric energy over moderate distances, from a number of $750-\mathrm{kw}.4400$-volt 60 -cycle three-phase generators. The station is to be increased in size by the installation of some larger modern three-phase generators, and from this station 6000 kw . are to be transmitted over a long distance transmission line at 44,000 volts. The transmission line has a length of 60 miles, and consists of three wires No. 0 B . & S. with 5 ft . between the wires.

The question arises, whether during times of light load the old $750-\mathrm{kw}$. generators can be used economically on the transmission line. These old machines give an electromotive force wave, which, like that of most earlier machines, differs considerably from a sine wave, and it is to be investigated, whether, due to this wave-shape distortion, the charging current of the transmission line will be so greatly increased over the value which it would have with a sine wave of voltage, as to make the use of these machines on the transmission line uneconomical or even unsafe.

Oscillograms of these machines, resolved into a trigonometric series, give for the voltage between each terminal and the neutral, or the Y voltage of the three-phase system, the equation:

$$\begin{aligned} e = e_0 \{ & \sin\theta - 0.12\sin(3\theta - 2.3^\circ) - 0.23\sin(5\theta - 1.5^\circ) \\ & + 0.13\sin(7\theta - 6.2^\circ)\}. \end{aligned} \tag{1}$$

In first approximation, the line capacity may be considered as a condenser shunted across the middle of the line; that is, half the line resistance and half the line reactance is in series with the line capacity.

As the receiving apparatus do not utilize the higher harmonics of the generator wave, when using the old generators, their voltage has to be transformed up so as to give the first harmonic or fundamental of 44,000 volts.

44,000 volts between the lines (or delta) gives $44,000 \div \sqrt{3} =$ 25,400 volts between line and neutral. This is the effective
value, and the maximum value of the fundamental voltage wave thus is: $25,400 \times \sqrt{2} = 36,000$ volts, or 36 kv .; that is, $e_0 = 36$, and

$$\begin{aligned} e = 36 \{ & \sin\theta - 0.12\sin(3\theta - 2.3^\circ) - 0.23\sin(5\theta - 1.5^\circ) \\ & + 0.13\sin(7\theta - 6.2^\circ)\}, \end{aligned} \tag{2}$$

would be the voltage supplied to the transmission line at the high potential terminals of the step-up transformers.

From the wire tables, the resistance per mile of No. 0 B. & S. copper line wire is $r_0 = 0.52\mathrm{ohm}$.

The inductance per mile of wire is given by the formula:

$$L_0 = 0.7415\log\frac{l_s}{l_r} + 0.0805\mathrm{mh}, \quad \ldots . \tag{3}$$

where l_s is the distance between the wires, and l_r the radius of the wire.

In the present case, this gives $l_s = 5\mathrm{ft.} = 60\mathrm{in.} \quad l_r = 0.1625\mathrm{in.}$ $L_0 = 1.9655\mathrm{mh.}$, and, herefrom it follows that the reactance, at $f = 60$ cycles is

$$x_0 = 2\pi f L_0 = 0.75 \text{ ohms per mile.} \ldots . \tag{4}$$

The capacity per mile of wire is given by the formula:

$$C_0 = \frac{0.0408}{\log \frac{l_8}{l_r}} \text{mf.} ; \ldots . \tag{5}$$

hence, in the present case, $C_0 = 0.0159\text{mf.}$, and the condensive reactance is derived herefrom as:

$$x_{c_0} = \frac{1}{2\pi f C_0} = 166000 \text{ohms}; \tag{6}$$

60 miles of line then give the condensive reactance,

$$x_c = \frac{x_{c_0}}{60} = 2770 \text{ ohms };$$

30 miles, or half the line (from the generating station to the middle of the line, where the line capacity is represented by a shunted condenser) give: the resistance, $r = 30r_0 = 15.6$ ohms ;
the inductive reactance, $x = 30x_0 = 22.5$ ohms, and the equivalent circuit of the line now consists of the resistance r, inductive reactance x and condensive reactance x_c, in series with each other in the circuit of the supply voltage e.
95. If $i =$ current in the line (charging current) the voltage consumed by the line resistance r is ri.

The voltage consumed by the inductive reactance x is $x\frac{di}{d\theta}$; the voltage consumed by the condensive reactance x_c is $x_c \int i d\theta$, and therefore,

$$e = x\frac{di}{d0} + ri + x_c \int i d0 \tag{7}$$

Differentiating this equation, for the purpose of eliminating the integral, gives
or

$$\frac{de}{d\theta} = x\frac{d^2 i}{d\theta^2} + r\frac{di}{d\theta} + x_c i$$

$$\frac{de}{d\theta} = 22.5\frac{d^2 i}{d\theta^2} + 16.6\frac{di}{d\theta} + 2770i. \tag{8}$$

The voltage e is given by (2), which equation, by resolving the trigonometric functions, gives

$$\begin{aligned} e = 36\sin\theta - 4.32\sin 30 - 8.28\sin 50 + 4.64\sin 7\theta \\ + 0.18\cos 30 + 0.22\cos 50 - 0.50\cos 7\theta; \end{aligned} \tag{9}$$

hence, differentiating,

$$\begin{aligned} \tfrac{de}{d\theta} = 36\cos\theta - 12.96\cos 30 - 41.4\cos 5\theta + 32.5\cos 7\theta \\ -0.54\sin 30 - 1.1\sin 50 + 3.5\sin 7\theta. \end{aligned} \tag{10}$$

Assuming now for the current i a trigonometric series with indeterminate coefficients,

$$\begin{aligned} i = a_1\cos\theta + a_3\cos 30 + a_5\cos 50 + a_7\cos 70 \\ + b_1\sin 0 + b_3\sin 30 + b_5\sin 50 + b_7\sin 7\theta, \end{aligned} \tag{11}$$

substitution of (10) and (11) into equation (8) must give an identity, from which equations for the determination of a_n and b_n are derived; that is, since the product of substitution must be an identity, all the factors of $\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta$, etc., must vanish, and this gives the eight equations:

$$\left.\begin{aligned} 36 &= 2770a_1 + 15.6b_1 - 22.5a_1; \\ 0 &= 2770b_1 - 15.6a_1 - 22.5b_1; \\ -12.96 &= 2770a_3 + 46.8b_3 - 202.5a_3; \\ -0.54 &= 2770b_3 - 46.8a_3 - 202.5b_3; \\ -41.4 &= 2770a_5 + 78b_5 - 562.5a_5; \\ -1.1 &= 2770b_5 - 78a_5 - 56.25b_5; \\ 32.5 &= 2770a_7 + 109.2b_7 - 1102.5a_7; \\ 3.5 &= 2770b_7 - 109.2a_7 - 1102.5b_7. \end{aligned}\right\}. \tag{12}$$

Resolved, these equations give

$$\left.\begin{aligned}a_1 &= 13.12;\\ b_1 &= 0.07;\\ a_3 &= -5.03;\\ b_3 &= -0.30;\\ a_5 &= -18.72;\\ b_5 &= -1.15;\\ a_7 &= 19.30;\\ b_7 &= 3.37;\end{aligned}\right\}\ldots\ldots. \qquad (14)$$

hence,
96. The effective value of this current is given as the square root of the sum of squares of the effective values of the individual harmonics, thus:

$$I = \sqrt{\sum \frac{a^2}{2} + \sum \frac{b^2}{2}} = 21.6 \text{amp.}$$

As the voltage between line and neutral is 25,400 effective, this gives $Q = 25,400 \times 21.6 = 540,000$ volt-amperes, or 540 kv .amp. per line, thus a total of $3Q = 1620\text{kv.}$-amp. charging current of the transmission line, when using the e.m.f. wave of these old generators.

It thus would require a minimum of 3 of the $750-\text{kw}$. generators to keep the voltage on the line, even if no power whatever is delivered from the line.

If the supply voltage of the transmission line were a perfect sine wave, it would, at 44,000 volts between the lines, be given by

$$e_1 = 36 \sin\theta, \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

which is the fundamental, or first harmonic, of equation (9).
Then the current i would also be a sine wave, and would be given by

$$\left.\begin{aligned}i_1 \quad &= a_1 \cos\theta + b_1 \sin\theta\\ &= 13.12 \cos\theta + 0.07 \sin\theta\\ &= 13.12 \cos(\theta - 0.3^\circ)\end{aligned}\right\} \qquad (16)$$

and its effective value would be

$$I_1 = \frac{13.12}{\sqrt{2}} = 9.3 \text{amp.} \quad . \quad . \quad . \qquad (17)$$

This would correspond to a kv.-amp. input to the line

$$3Q_1 = 3 \times 25.4 \times 9.3 = 710 \text{kv.-amp .}$$

The distortion of the voltage wave, as given by equation (1), thus increases the charging volt-amperes of the line from 710
kv.-amp. to $1620\text{kv.} - \text{amp.}$ or 2.28 times, and while with a sine wave of voltage, one of the $750-\text{kw}$. generators would easily be able to supply the charging current of the line, due to the

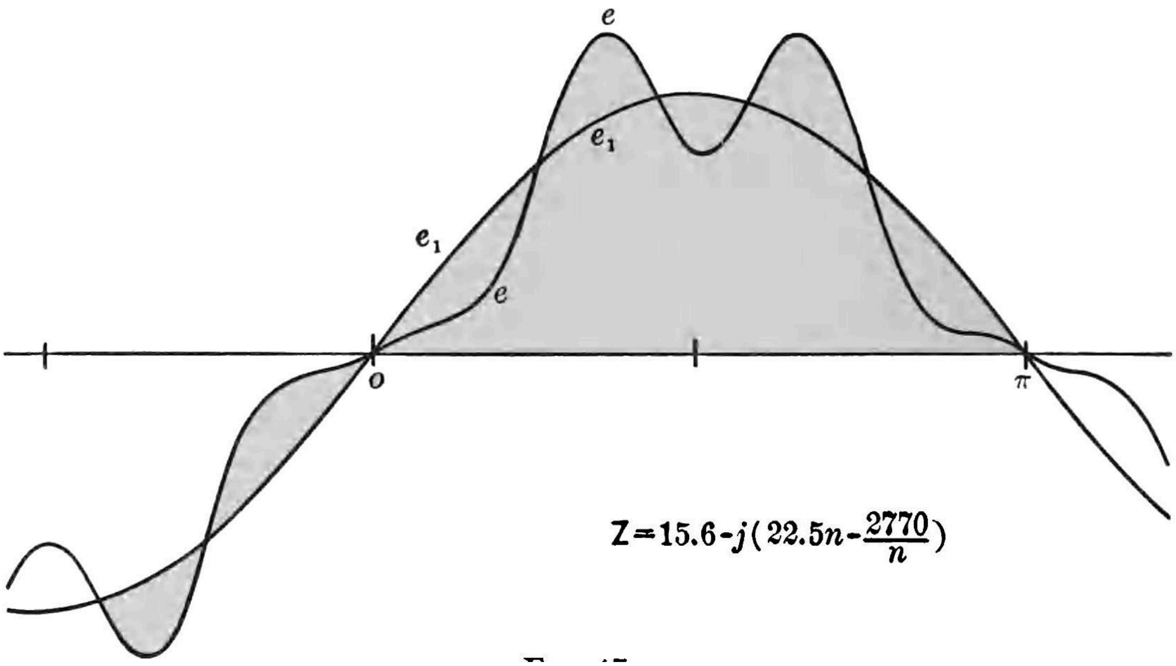

Fig. 47.

wave shape distortion, more than two generators are required. It would, therefore, not be economical to use these generators on the transmission line, if they can be used for any other purposes, as short-distance distribution.

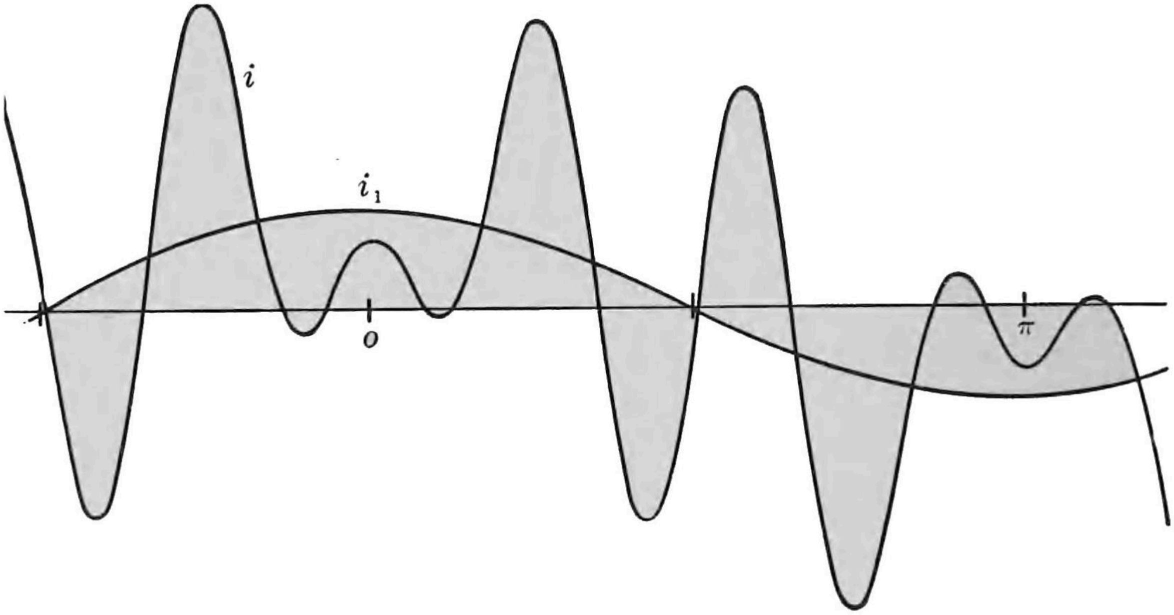

Fig. 48.

In Figs. 47 and 48 are plotted the voltage wave and the current wave, from equations (9) and (14) respectively, and the numerical values, from 10 deg . to 10 deg ., recorded in Table XII.

In Figs. 47 and 48 the fundamental sine wave of voltage and current are also shown. As seen, the distortion of current is enormous, and the higher harmonics predominate over the fundamental. Such waves are occasionally observed as charging currents of transmission lines or cable systems.
97. Assuming now that a reactive coil is inserted in series with the transmission line, between the step-up transformers and the line, what will be the voltage at the terminals of this reactive coil, with the distorted wave of charging current traversing the reactive coil, and how does it compare with the voltage existing with a sine wave of charging current?

Let L = inductance, thus $x = 2\pi fL$ = reactance of the coil, and neglecting its resistance, the voltage at the terminals of the reactive coil is given by

$$e' = -x\frac{di}{d\theta}\dots\dots\dots\dots \tag{18}$$

Substituting herein the equation of current, (11), gives

$$\left.\begin{array}{l} e' = x\left\{a_1 \sin\theta + 3a_3 \sin 3\theta + 5a_5 \sin 5\theta + 7a_7 \sin 7\theta \right. \\ \left. -b_1 \cos\theta - 3b_3 \cos 3\theta - 5b_5 \cos 5\theta - 7b_7 \cos 7\theta\right\}; \end{array}\right\}. \tag{19}$$

hence, substituting the numerical values (13),

$$\left.\begin{array}{rl} e' = & x\{13.12 \sin\theta - 15.09 \sin 3\theta - 93.6 \sin 5\theta + 135.1 \sin 70 \\ & -0.07 \cos\theta + 0.90 \cos 3\theta + 5.75 \cos 5\theta - 23.6 \cos 70\} \\ = & x\left\{13.12 \sin(\theta - 0.3^\circ) - 15.12 \sin(3\theta - 3.3^\circ)\right. \\ & \left. -93.8 \sin(5\theta - 3.6^\circ) + 139.1 \sin(7\theta - 9.9^\circ)\right\} \end{array}\right\} \tag{20}$$

This voltage gives the effective value

$$E' = x\sqrt{\frac{1}{2}\left\{13.12^2 + 15.12^2 + 93.8^2 + 139.1^2\right\}} = 119.4x,$$

while the effective value with a sine wave would be from (17),

$$E'_1 = xI_1 = 9.3x$$

hence, the voltage across the reactance x has been increased 12.8 times by the wave distortion.

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*The numerical values of temperature cannot claim any great absolute accuracy, as they are averaged over a relatively small number of years only, and observed by instruments of only moderate accuracy. For the purpose of illustrating the resolution of the empirical curve into a trigonometric series, this is not essential, however.